REDUCTIVE SHAFAREVICH CONJECTURE

by

Ya Deng & Katsutoshi Yamanoi

With an appendix joint with Ludmil Katzarkov

Abstract. — In this paper, we present a more accessible proof of Eyssidieux's proof of the reductive Shafarevich conjecture in 2004, along with several generalizations. In a nutshell, we prove the holomorphic convexity of the covering of a projective normal variety X, which corresponds to the intersection of kernels of reductive representations $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. Our approach avoids the necessity of using the reduction mod p method employed in Eyssidieux's original proof. Moreover, we extend the theorems to singular normal varieties under a weaker condition of absolutely constructible subsets, thereby answering a question by Eyssidieux, Katzarkov, Pantev, and Ramachandran. Additionally, we construct the Shafarevich morphism for reductive representations over quasi-projective varieties unconditionally, and proving its algebraic nature at the function field level.

Contents

Key words and phrases. — Reductive Shafarevich conjecture, Hamonic mapping to Bruhat-Tits buildings, period mappings and period domain, variation of Hodge structures, Holomorphic convexity.

0. Introduction

0.1. Shafarevich conjecture. — In his famous textbook "Basic Algebraic Geometry" [Sha77, p 407], Shafarevich raised the following tantalizing conjecture.

***Conjecture 0.1* (Shafarevich). —** *Let X be a complex projective variety. Then its universal covering is holomorphically convex.*

Recall that a complex normal space X is *holomorphically convex* if it satisfies the following condition: for each compact $K \subset X$, its *holomorphic hull*

$$\left\{ x \in X \mid |f(x)| \leq \sup_K |f|, \forall f \in \mathcal{O}(X) \right\},$$

is compact. X is *Stein* if it is holomorphically convex and holomorphically separable, i.e. for distinct x and y in X, there exists $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$. By the Cartan-Remmert theorem, a complex space X is holomorphically convex if and only if it admits a proper surjective holomorphic mapping ϕ onto some Stein space.

The study of Conjecture 0.1 for smooth projective surfaces has been a subject of extensive research since the mid-1980s. Gurjar-Shastri [GS85] and Napier [Nap90] initiated this investigation, while Kollár [Kol93] and Campana [Cam94] independently explored the conjecture in the 1990s, employing the tools of Hilbert schemes and Barlet cycle spaces. In 1994, Katzarkov discovered that non-abelian Hodge theories developed by Simpson [Sim92] and Gromov-Schoen [GS92] can be utilized to prove Conjecture 0.1. His initial work [Kat97] demonstrated Conjecture 0.1 for projective varieties with nilpotent fundamental groups. Shortly thereafter, he and Ramachandran [KR98] successfully established Conjecture 0.1 for smooth projective surfaces whose fundamental groups admit a faithful Zariski-dense representation in a reductive complex algebraic group. Building upon the ideas presented in [KR98] and [Mok92], Eyssidieux further developed non-abelian Hodge theoretic arguments in higher dimensions. In [Eys04] he proved that Conjecture 0.1 holds for any *smooth* projective variety whose fundamental group possesses a faithful representation that is Zariski dense in a reductive complex algebraic group. This result is commonly referred to as the "*Reductive Shafarevich conjecture*". It is worth emphasizing that the work of Eyssidieux [Eys04] is not only ingenious but also highly significant in subsequent research. It serves as a foundational basis for advancements in the linear Shafarevich conjecture [EKPR12] and the exploration of compact Kähler cases [CCE15]. More recently, there have been significant advancements in the quasi-projective setting by Green-Griffiths-Katzarkov [GGK22] and Aguilar-Campana [AC23], particularly when considering the case of nilpotent fundamental groups.

0.2. Main theorems. — The aim of this paper is to present a more comprehensive and complete proof of Eyssidieux's results on the reductive Shafarevich conjecture and its associated problems, as originally discussed in [Eys04]. Additionally, we aim to extend these results to the cases of quasi-projective and singular varieties. Our first main result is the *unconditional* construction of the *Shafarevich morphism* for reductive representations. Additionally, we establish the algebraicity of the Shafarevich morphism at the function field level.

***Theorem A* (=Theorems 3.39 and 3.46).** — *Let X be a quasi-projective normal variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. Then*

(i) *there exists a dominant holomorphic map $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ to a complex normal space $\mathrm{Sh}_\varrho(X)$ whose general fibers are connected such that for any closed subvariety $Z \subset X$, $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite if and only if $\mathrm{sh}_\varrho(Z)$ is a point.*

Furthermore, when X is a smooth, after we replace X by some finite étale cover and ϱ by its pullback over the cover, there exists another smooth quasi-projective variety X' containing X as a Zariski dense open subset such that:

(ii) *ϱ extends to a reductive representation $\varrho_0 : \pi_1(X') \to \mathrm{GL}_N(\mathbb{C})$;*
(iii) *the Shafarevich morphism $\mathrm{sh}_{\varrho_0} : X' \to \mathrm{Sh}_{\varrho_0}(X')$ exists, which is a holomorphic proper fibration;*
(iv) *$\mathrm{sh}_\varrho = \mathrm{sh}_{\varrho_0}|_X$; namely, we have the following commutative diagram:*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\mathrm{sh}_\varrho"'] & X' \arrow[d, "\mathrm{sh}_{\varrho_0}"] \\ \mathrm{Sh}_\varrho(X) \arrow[r, equal] & \mathrm{Sh}_{\varrho_0}(X') \end{tikzcd}$$

(v) *There exists a bimeromorphic map $h : \mathrm{Sh}_\varrho(X) \dashrightarrow Y$ to a quasi-projective normal variety Y.*
(vi) *The composition $h \circ \mathrm{sh}_\varrho : X \dashrightarrow Y$ is a rational map.*

The holomorphic map $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ that satisfies the properties in Theorem A.(i) will be called the *Shafarevich morphism* of ϱ.

The proof of Theorem A.(i) relies on a more technical result but with richer information, cf. Theorem 3.20.

Remark 0.2. — It is noticeable that Theorems A.(i) to A.(iv) extend the previous theorems of Griffiths [Gri70] when ϱ underlies a $\mathbb{Z}$-variation of Hodge structures. In this case, the representation ϱ_0 in Theorem A.(ii) is constructed in [Gri70, Theorem 9.5]. Additionally, Griffiths proved in [Gri70, Theorem 9.6] that the period mapping $p : X' \to \mathscr{D}/\Gamma$ associated with ϱ_0 is proper, where $\mathscr{D}$ represents the period domain of the $\mathbb{C}$-VHS, and Γ denotes the monodromy group of ϱ_0. It can be easily verified that the Shafarevich morphism $\mathrm{sh}_{\varrho_0} : X' \to \mathrm{Sh}_{\varrho_0}(X')$ corresponds to the Stein factorization of the period mapping $p : X' \to \mathscr{D}/\Gamma$, and that Theorem A.(iv) holds.

We conjecture that $\mathrm{Sh}_{\varrho_0}(X')$ is quasi-projective and sh_{ϱ_0} is an algebraic morphism (cf. Conjecture 3.44). Our conjecture is motivated by Griffiths' conjecture, which predicted the same result when ϱ underlies a $\mathbb{Z}$-VHS. Consequently, we can interpret the results presented in Theorems A.(v) and A.(vi) as supporting evidence for our conjecture at the function field level. It is worth noting that Sommese proved Theorems A.(v) and A.(vi) when ϱ underlies a $\mathbb{Z}$-VHS in [Som78], utilizing L^2-methods. We adopt the same approach in [Som78] to prove Theorems A.(v) and A.(vi). Griffiths' conjecture was recently proved by Baker-Brunebarbe-Tsimerman [BBT23] using o-minimal geometry.

Our second main result focuses on the holomorphic convexity of topological Galois coverings associated with reductive representations of fundamental groups within *absolutely constructible subsets* of character varieties $M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)$, where X represents a projective normal variety.

Theorem B **(=Theorems 4.25 and A.3). —** *Let X be a projective normal variety, and let $\mathfrak{C}$ be an absolutely constructible subset of $M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$ as defined in Definitions 1.17 and A.1. We assume that $\mathfrak{C}$ is defined on $\mathbb{Q}$. Set $H := \cap_{\varrho} \ker \varrho$, where $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations such that $[\varrho] \in \mathfrak{C}(\mathbb{C})$. Let $\widetilde{X}$ be the universal covering of X, and denote $\widetilde{X}_{\mathfrak{C}} := \widetilde{X}/H$. Then the complex space $\widetilde{X}_{\mathfrak{C}}$ is holomorphically convex. In particular, we have*

(i) *the covering of X corresponding to the intersections of the kernels of all reductive representations of $\pi_1(X)$ in $\mathrm{GL}_N(\mathbb{C})$ is holomorphically convex;*
(ii) *if $\pi_1(X)$ is a subgroup of $\mathrm{GL}_N(\mathbb{C})$ whose Zariski closure is reductive, then the universal covering $\widetilde{X}$ of X is holomorphically convex.*

For large representations, we have the following result.

Theorem C **(=Theorems 4.26 and A.5). —** *Let X and $\mathfrak{C}$ be as described in Theorem B. If $\mathfrak{C}$ is* large, *meaning that for any closed subvariety Z of X, there exists a reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in \mathfrak{C}(\mathbb{C})$ and $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite, then all intermediate coverings of X between $\widetilde{X}$ and $\widetilde{X}_{\mathfrak{C}}$ are Stein spaces.*

In addition to employing new methods for the proof of Theorems B and C, it yields a stronger result compared to [Eys04] in two aspects:

(a) The definition of absolutely constructible subsets (cf. Definitions 1.17 and A.1) in our proof is more general than the one provided in [Eys04]. Our definition allows for a broader range of applications, including the potential extension of Conjecture 0.1 to quasi-projective varieties, which is currently our ongoing project.
(b) Our result extends to the case where X is a singular variety, whereas in [Eys04], the result is limited to smooth varieties. This expansion of our result answers a question raised by Eyssidieux, Katzarkov, Pantev and Ramachadran in their celebrated work on linear Shafarevich conjecture for smooth projective varieties (cf. [EKPR12, p. 1549]).

We remark that Theorem C is not a direct consequence of Theorem B. It is important to note that Theorem C holds significant practical value in the context of singular varieties. Indeed, finding a large representation over a *smooth* projective variety can be quite difficult. In practice, the usual approach involves constructing large representations using the Shafarevich morphism in Theorem A, resulting in large representations of fundamental groups of *singular normal* varieties. Therefore, the extension of Theorem C to singular varieties allows for more practical applicability.

0.3. Comparison with [Eys04] and Novelty. — It is worth noting that Eyssidieux [Eys04] does not explicitly require absolutely constructible subsets $\mathfrak{C}$ to be defined over $\mathbb{Q}$, although it may seem to be an essential condition (cf. Remark 3.13). Regarding Theorem A, it represents a new result that significantly builds upon our previous work [CDY22]. While Theorem C is not explicitly stated in [Eys04], it should be possible to derive it for smooth projective varieties X based on the proof provided therein. However, it is worth noting that the original proof in [Eys04] is known for its notoriously difficult and involved nature, with certain aspects outlined without sufficient detail. One of the main goals of this paper is to provide a relatively accessible proof for Theorem B by incorporating more detailed explanations. We draw inspiration from some of the methods introduced in our recent work [CDY22], which aids in presenting a more comprehensible proof. Our proofs of Theorems B and C require us to apply Eyssidieux's Lefschetz theorem from [Eys04]. We also owe many ideas to Eyssidieux's work in [Eys04] and frequently draw upon them without explicit citation.

Despite this debt, there are some novelties in our approach, including:

— An avoidance of the reduction mod p method used in [Eys04].
— A new and more canonical construction of the Shafarevich morphism that incorporates both rigid and non-rigid cases, previously treated separately in [Eys04].
— The construction of the Shafarevich morphism for reductive representations over quasi-projective varieties, along with a proof of its algebraic property at the function field level.
— A detailed exposition of the application of Simpson's absolutely constructible subsets to the proof of holomorphic convexity in Theorems B and C (cf. § 4.1 and Theorem 4.21). This application was briefly outlined in [Eys04, Proof of Proposition 5.4.6], but we present a more comprehensive approach, providing complete details.

The main part of this paper was completed in February 2023 and was subsequently shared with several experts in the field in April for feedback. During the revision process, it came to our attention that Brunebarbe [Bru23] recently announced a result similar to Theorem A.(i). In [Bru23, Theorem B] Brunebarbe claims the existence of the Shafarevich morphism under a stronger assumption of infinite monodromy at infinity and torsion-freeness of the representation, and he does not address the algebraicity of Shafarevich morphisms. It seems that some crucial aspects of the arguments in [Bru23] need to be carefully addressed, particularly those related to non-abelian Hodge theories may have been overlooked (cf. Remark 3.36).

Convention and notation. — In this paper, we use the following conventions and notations:

— Quasi-projective varieties and their closed subvarieties are assumed to be positive-dimensional and irreducible unless specifically mentioned otherwise. Zariski closed subsets, however, may be reducible.
— Fundamental groups are always referred to as topological fundamental groups.
— If X is a complex space, its normalization is denoted by X^{norm}.
— The bold letter Greek letter $\boldsymbol{\varrho}$ (or $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$...) denotes a family of finite reductive representations $\{\varrho_i : \pi_1(X) \to \mathrm{GL}_N(K)\}_{i=1,\ldots,k}$ where K is some non-archimedean local field or complex number field.
— A *proper holomorphic fibration* between complex spaces $f : X \to Y$ is *surjective* and each fiber of f is *connected*.
— Let X be a compact normal Kähler space and let $V \subset H^0(X, \Omega_X^1)$ be a $\mathbb{C}$-linear subspace. The *generic rank* of V is the largest integer r such that $\mathrm{Im}[\Lambda^r V \to H^0(X, \Omega_X^r)] \neq 0$.
— For a quasi-projective normal variety X, we denote by $M_{\mathrm{B}}(X, N)$ the character variety of the representations of $\pi_1(X)$ into GL_N. For any linear representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ where K is some extension of $\mathbb{Q}$, we denote by $[\varrho] \in M_{\mathrm{B}}(X, N)(K)$ the equivalent class of ϱ.
— $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

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1. Technical preliminary

1.1. Admissible coordinates. — The following definition of *admissible coordinates* introduced in [Moc07a] will be used throughout the paper.

Definition 1.1. — (Admissible coordinates) Let X be a complex manifold and let D be a simple normal crossing divisor. Let x be a point of D, and assume that $\{D_j\}_{j=1,\ldots,\ell}$ be components of D containing x. An *admissible coordinate* centered at x is the tuple $(U; z_1, \ldots, z_n; \varphi)$ (or simply $(U; z_1, \ldots, z_n)$ if no confusion arises) where

— U is an open subset of X containing x.
— there is a holomorphic isomorphism $\varphi : U \to \mathbb{D}^n$ so that $\varphi(D_j) = (z_j = 0)$ for any $j = 1, \ldots, \ell$.

1.2. Tame and pure imaginary harmonic bundles. — Let $\overline{X}$ be a compact complex manifold, $D = \sum_{i=1}^{\ell} D_i$ be a simple normal crossing divisor, $X = \overline{X}\backslash D$ be the complement of D and $j : X \to \overline{X}$ be the inclusion.

***Definition 1.2* (Higgs bundle).** — A *Higgs bundle* on X is a pair (E, θ) where E is a holomorphic vector bundle, and $\theta : E \to E \otimes \Omega_X^1$ is a holomorphic one form with value in $\mathrm{End}(E)$, called the *Higgs field*, satisfying $\theta \wedge \theta = 0$.

Let (E, θ) be a Higgs bundle over a complex manifold X. Suppose that h is a smooth hermitian metric of E. Denote by ∇_h the Chern connection of (E, h), and by $\theta_h^\dagger$ the adjoint of θ with respect to h. We write $\theta^\dagger$ for $\theta_h^\dagger$ for short if no confusion arises. The metric h is *harmonic* if the connection $\nabla_h + \theta + \theta^\dagger$ is flat.

***Definition 1.3* (Harmonic bundle).** — A harmonic bundle on X is a Higgs bundle (E, θ) endowed with a harmonic metric h.

Let (E, θ, h) be a harmonic bundle on X. Let p be any point of D, and $(U; z_1, \ldots, z_n)$ be an admissible coordinate centered at p. On U, we have the description:

$$\theta = \sum_{j=1}^{\ell} f_j d\log z_j + \sum_{k=\ell+1}^{n} f_k dz_k. \tag{1.1}$$

***Definition 1.4* (Tameness).** — Let t be a formal variable. For any $j = 1, \ldots, \ell$, the characteristic polynomial $\det(f_j - t) \in \mathcal{O}(U\backslash D)[t]$, is a polynomial in t whose coefficients are holomorphic functions. If those functions can be extended to the holomorphic functions over U for all j, then the harmonic bundle is called *tame* at p. A harmonic bundle is *tame* if it is tame at each point.

For a tame harmonic bundle (E, θ, h) over $\overline{X}\backslash D$, we prolong E over $\overline{X}$ by a sheaf of $\mathcal{O}_{\overline{X}}$-module ${}^\diamond E_h$ as follows:

$${}^\diamond E_h(U) = \{\sigma \in \Gamma(U\backslash D, E|_{U\backslash D}) \mid |\sigma|_h \lesssim \prod_{i=1}^{\ell} |z_i|^{-\varepsilon} \ \text{ for all } \varepsilon > 0\}.$$

In [Moc07a] Mochizuki proved that ${}^\diamond E_h$ is locally free and that θ extends to a morphism

$${}^\diamond E_h \to {}^\diamond E_h \otimes \Omega_{\overline{X}}^1(\log D),$$

which we still denote by θ.

***Definition 1.5* (Pure imaginary).** — Let (E, h, θ) be a tame harmonic bundle on $\overline{X}\backslash D$. The residue $\mathrm{Res}_{D_i}\theta$ induces an endomorphism of ${}^\diamond E_h|_{D_i}$. Its characteristic polynomial has constant coefficients, and thus the eigenvalues are all constant. We say that (E, θ, h) is *pure imaginary* if for each component D_j of D, the eigenvalues of $\mathrm{Res}_{D_i}\theta$ are all pure imaginary.

One can verify that Definition 1.5 does not depend on the compactification $\overline{X}$ of $\overline{X}\backslash D$.

***Theorem 1.6* (Mochizuki [Moc07b, Theorem 25.21]).** — *Let $\overline{X}$ be a projective manifold and let D be a simple normal crossing divisor on $\overline{X}$. Let (E, θ, h) be a tame pure imaginary harmonic bundle on $X := \overline{X} \backslash D$. Then the flat bundle $(E, D_h + \theta + \theta^\dagger)$ is semi-simple. Conversely, if (V, ∇) is a semisimple flat bundle on X, then there is a tame pure imaginary harmonic bundle (E, θ, h) on X so that $(E, \nabla_h + \theta + \theta^\dagger) \simeq (V, \nabla)$. Moreover, when ∇ is* simple, *then any such harmonic metric h is unique up to positive multiplication.*

The following important theorem by Mochizuki will be used throughout the paper.

Theorem 1.7. — *where Y is smooth and X is normal. For any reductive representation $\varrho : \pi_1(Y) \to \mathrm{GL}_N(K)$, where K is a non-archimedean local field of characteristic zero or a complex number field, the pullback $f^*\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ is also reductive.*

Proof. — If K is a non-archimedean local field of characteristic zero, then there is an abstract embedding $K \hookrightarrow \mathbb{C}$. Therefore, it suffices to prove the theorem for $K = \mathbb{C}$.

Let $\mu : X' \to X$ be a desingularization of X. By [Moc07b, Theorem 25.30], $(f \circ \mu)^*\varrho : \pi_1(X') \to \mathrm{GL}_N(\mathbb{C})$ is reductive. Since $\mu_* : \pi_1(X') \to \pi_1(X)$ is surjective, it follows that $(f \circ \mu)^*\varrho(\pi_1(X')) = f^*\varrho(\pi_1(X))$. Hence, $f^*\varrho$ is also reductive. □

1.3. Positive currents on normal complex spaces. — For this subsection, we refer to [Dem85] for more details.

Definition 1.8. — Let Z be an irreducible normal complex space. A upper semi continuous function $\phi : Z \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic if it is not identically $-\infty$ and every point $z \in Z$ has a neighborhood U embeddable as a closed subvariety of the unit ball B of some $\mathbb{C}^M$ in such a way that $\phi|_U$ extends to a psh function on B.

A closed positive current with continuous potentials ω on Z is specified by a data $\{U_i, \phi_i\}_i$ of an open covering $\{U_i\}_i$ of Z, a continuous psh function ϕ_i defined on U_i such that $\phi_i - \phi_j$ is pluriharmonic on $U_i \cap U_j$.

A closed positive current with continuous potentials Z is a Kähler form iff its local potentials can be chosen smooth and strongly plurisubharmonic.

A psh function ϕ on Z is said to satisfy $\mathrm{dd^c}\phi \geq \omega$ iff $\phi - \phi_i$ is psh on U_i for every i.

In other words, a closed positive current with continuous potentials is a section of the sheaf $C^0 \cap PSH_Z / \operatorname{Re}(O_Z)$.

Definition 1.9. — Assume Z to be compact. The class of a closed positive current with continuous potentials is its image in $H^1(Z, \operatorname{Re}(O_Z))$.

A class in $H^1(Z, \operatorname{Re}(O_Z))$ is said to be Kähler if it is the image of a Kähler form.

To make contact with the usual terminology observe that if Z is a compact Kähler manifold $H^1(Z, \operatorname{Re}(O_Z)) = H^{1,1}(Z, \mathbb{R})$. Hence we use abuse of notation to write $H^{1,1}(Z, \mathbb{R})$ instead of $H^1(Z, \operatorname{Re}(O_Z))$ in this paper.

Lemma 1.10. — *Let $f : X \to Y$ be a Galois cover with Galois group G, where X and Y are both irreducible normal complex space. Let T be a positive $(1,1)$-current on X with continuous potential. Assume that T is invariant under G. Then there is a closed positive $(1,1)$-current S on Y with continuous potential such that $T = f^*S$.*

Proof. — Since the statement is local, we may assume that $T = \mathrm{dd^c}\varphi$ such that $\varphi \in C^0(X)$. Define a function on Y by

$$f_*\varphi(y) := \sum_{x \in f^{-1}(y)} \varphi(x)$$

here the sums are counted with multiplicity. By [Dem85, Proposition 1.13.(b)], we know that $f_*\varphi$ is a psh function on Y and

$$\mathrm{dd^c} f_*\varphi = f_*T.$$

One can see that $f_*\varphi$ is also continuous. Define a current $S := \frac{1}{\deg f} f_* T$. Since T is G-invariant, it follows that $f^*S = T$ outside the branch locus of f. Since $f^*S = \frac{1}{\deg f}\mathrm{dd^c}(f_*\varphi)\circ f$, the potential of f^*S is continuous. It follows that $f^*S = T$ over the whole X. □

1.4. Holomorphic forms on complex normal spaces. — There are many ways to define holomorphic forms on complex normal spaces. For our purpose of the current paper, we use the following definition in [Fer70].

Definition 1.11. — Let X be a normal complex space. Let $(A_i)_{i\in I}$ be an open finite covering of X such that each subset A_i is an analytic subset of some open subset $\Omega_i \subset \mathbb{C}^{N_i}$. The space of holomorphic p-forms, denoted by Ω_X^p, is defined by local restrictions of holomorphic p-forms on the sets Ω_i above to A_i^{reg}, where A_i^{reg} is the smooth locus of A_i.

The following fact will be used throughout the paper.

Lemma 1.12. — *Let $f : X \to Y$ be a holomorphic map between normal complex spaces. Then for any holomorphic p-form ω on Y, $f^*\omega$ a holomorphic p-form on Y.*

Proof. — By Definition 1.11, for any $x \in X$, there exist

— a neighborhood A (resp. B) of x (resp. $f(x)$) such that A (resp. B) is an analytic subset of some open $\Omega \subset \mathbb{C}^m$ (resp. $\Omega' \subset \mathbb{C}^n$).
— a holomorphic map $\tilde{f} : \Omega \to \Omega'$ such that $\tilde{f}|_A = f|_A$.
— A holomorphic p-form $\tilde{\omega}$ on Ω' such that $\omega = \tilde{\omega}|_B$.

Therefore, we can define $f^*\omega|_A := \tilde{f}^*\tilde{\omega}|_\Omega$. One can check that this does not depend on the choice of local embeddings of X and Y. □

1.5. The criterion for Kähler classes. — We will need the following extension of the celebrated Demailly-Păun's theorem [DP04] on characterization of Kähler classes on complex normal Kähler spaces by Das-Hacon-Păun in [DHP22].

***Theorem 1.13* ([DHP22, Corollary 2.39]).** — *Let X be a compact normal Kähler space, ω a Kähler form on X, and $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$, then α is Kähler if and only if $\int_V \alpha^k \wedge \omega^{p-k} > 0$ for every analytic p-dimensional subvariety $V \subset X$ and for all $0 < k \leq p$.*

1.6. Some criterion for Stein space. — We require the following criterion for the Stein property of a topological Galois covering of a compact complex normal space.

***Proposition 1.14* ([Eys04, Proposition 4.1.1]).** — *Let X be a compact complex normal space and let $\pi : \widetilde{X}' \to X$ be some topological Galois covering. Let T be a positive current on X with continuous potential such that $\{T\}$ is a Kähler class. Assume that there exists a continuous plurisubharmonic function $\phi : \widetilde{X}' \to \mathbb{R}_{\geq 0}$ such that $\mathrm{dd^c}\phi \geq \pi^*T$. Then $\widetilde{X}'$ is a Stein space.*

1.7. Some facts on moduli spaces of rank 1 local systems. — For this subsection we refer the readers to [Sim93] for a systematic treatment. Let X be a smooth projective variety defined over a field $K \subset \mathbb{C}$. Let $M = M(X)$ denote the moduli space of complex local systems of rank one over X. We consider M as a real analytic group under the operation of tensor product. There are three natural algebraic groups M_{B}, M_{DR} and $\mathrm{M}_{\mathrm{Dol}}$ whose underlying real analytic groups are canonically isomorphic to M. The first is Betti moduli space $M_{\mathrm{B}} := \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$. The second is De Rham moduli space M_{DR} which consists of pairs (L, ∇) where L is a holomorphic line bundle on X and ∇ is an integrable algebraic connection on L. The last one M_{Dol} is moduli spaces of rank one Higgs bundles on X. Recall that $\mathrm{Pic}^\tau(X)$ denotes the group of line bundles on X whose first Chern classes are torsion. We have

$$M_{\mathrm{Dol}} = \mathrm{Pic}^\tau(X) \times H^0(X, \Omega_X^1)$$

For any subset $S \subset M$, let S_{B}, S_{Dol} and S_{DR} denote the corresponding subsets of M_{B}, M_{DR} and $\mathrm{M}_{\mathrm{Dol}}$.

***Definition 1.15* (Triple torus).** — A triple torus is a closed, connected real analytic subgroup $N \subset M$ such that $N_{\mathrm{B}}, N_{\mathrm{DR}}$, and N_{Dol} are algebraic subgroups defined over $\mathbb{C}$. We say that a closed real analytic subspace $S \subset M$ is a translate of a triple torus if there exists a triple torus $N \subset M$ and a point $v \in M$ such that $S = \{v \otimes w, w \in N\}$. Note that, in this case, any choice of $v \in M$ will do.

We say that a point $v \in M$ is torsion if there exists an integer $a > 0$ such that $v^{\otimes a} = 1$. Let M^{tor} denote the set of torsion points. Note that for a given integer a, there are only finitely many solutions of $v^{\otimes a} = 1$. Hence, the points of $M_{\mathrm{B}}^{\mathrm{tor}}$ are defined over $\overline{\mathbb{Q}}$, and the points of $M_{\mathrm{DR}}^{\mathrm{tor}}$ and $M_{\mathrm{Dol}}^{\mathrm{tor}}$ are defined over $\overline{K}$.

We say that a closed subspace S is a torsion translate of a triple torus if S is a translate of a triple torus N by an element $v \in M^{\mathrm{tor}}$. This is equivalent to asking that S be a translate of a triple torus, and contain a torsion point.

Let A be the Albanese variety of X (which can be defined as $H^0(X, \Omega_X^1)^*/H_1(X, \mathbb{Z})$). Let $X \to A$ be the map from X into A given by integration (from a basepoint, which will be suppressed in the notation but assumed to be defined over $\overline{K}$). Pullback of local systems gives a natural map from $M(A)$ to $M(X)$, which is an isomorphism

$$M(A) \cong M^0(X),$$

where $M^0(X)$ is the connected component of $M(X)$ containing the trivial rank one local system. The Albanese variety A is defined over $\overline{K}$. We recall the following result in [Sim93, Lemma 2.1].

***Lemma 1.16* (Simpson).** — *Let $N \subset M$ be a closed connected subgroup such that $N_{\mathrm{B}} \subset M_{\mathrm{B}}$ is complex analytic and $\mathrm{N}_{\mathrm{Dol}} \subset \mathrm{M}_{\mathrm{Dol}}$ is an algebraic subgroup. Then there is a connected abelian subvariety $P \subset A$, defined over $\overline{K}$, such that N is the image in M of $M(A/P)$. In particular, N is a triple torus in M.*

1.8. Absolutely constructible subsets. — In this section we will recall some facts on *absolutely constructible subsets* (resp. *absolutely closed subsets*) introduced by Simpson in [Sim93, §6] and later developed by Budur-Wang [WB20].

Let X be a smooth projective variety defined over a subfield ℓ of $\mathbb{C}$. Let G be a reductive group defined over $\bar{\mathbb{Q}}$. The representation scheme of $\pi_1(X)$ is an affine $\bar{\mathbb{Q}}$-algebraic scheme described by its functor of points:

$$R(X, G)(\operatorname{Spec} A) := \operatorname{Hom}(\pi_1(X), G(A))$$

for any $\bar{\mathbb{Q}}$-algebra A. The character scheme of $\pi_1(X)$ with values in G is the finite type affine scheme $M_{\mathrm{B}}(X, G) := R(X, G) /\!\!/ G$, where "$/\!\!/$" denotes the GIT quotient. If $G = \mathrm{GL}_N$, we simply write $M_{\mathrm{B}}(X, N) := M_{\mathrm{B}}(X, \mathrm{GL}_N)$. Simpson constructed a quasi-projective scheme $M_{\mathrm{DR}}(X, G)$, and $M_{\mathrm{Dol}}(X, G)$ over ℓ. The $\mathbb{C}$-points of $M_{\mathrm{DR}}(X, G)$ are in bijection with the equivalence classes of flat G-connections with reductive monodromy. There are natural isomorphisms

$$\psi : M_{\mathrm{B}}(X, G)(\mathbb{C}) \to M_{\mathrm{DR}}(X, G)(\mathbb{C})$$

such that ψ is an isomorphism of complex analytic spaces. For each automorphism $\sigma \in \operatorname{Aut}(\mathbb{C}/\mathbb{Q})$, let $X^\sigma := X \times_\sigma \mathbb{C}$ be the conjugate variety of X, which is also smooth projective. There is a natural map

$$p_\sigma : M_{\mathrm{DR}}(X, G) \to M_{\mathrm{DR}}\left(X^\sigma, G^\sigma\right).$$

Let us now introduce the following definition of absolutely constructible subsets.

***Definition 1.17* (Absolutely constructible subset).** — A subset $\mathfrak{C} \subset M_{\mathrm{B}}(X, G)(\mathbb{C})$ is an *absolutely constructible subset* (resp. *absolutely closed subset*) if the following conditions are satisfied.

(i) $\mathfrak{C}$ is the a $\bar{\mathbb{Q}}$-constructible (resp. $\bar{\mathbb{Q}}$-closed) subset of $M_{\mathrm{B}}(X, G)$.

(ii) For each $\sigma \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, there exists a $\bar{\mathbb{Q}}$-constructible (resp. $\bar{\mathbb{Q}}$-closed) set $\mathfrak{C}^\sigma \subset M_{\mathrm{B}}(X^\sigma, G^\sigma)(\mathbb{C})$ such that $\psi^{-1} \circ p_\sigma \circ \psi(\mathfrak{C}) = \mathfrak{C}^\sigma$.
(iii) $\mathfrak{C}(\mathbb{C})$ is preserved by the action of $\mathbb{R}^*$ defined in § 2.4.

Remark 1.18. — Note that this definition is significantly weaker than the notion of absolutely constructible sets defined in [Sim93, Eys04], as it does not consider moduli spaces of semistable Higgs bundles with trivial characteristic numbers, and it does not require that $\psi(\mathfrak{C})$ is $\bar{\mathbb{Q}}$-constructible in $M_{\mathrm{DR}}(X, G)(\mathbb{C})$. This revised definition allows for a broader range of applications, including quasi-projective varieties. In [Sim93, Eys04], the preservation of $\mathfrak{C}(\mathbb{C})$ under the action of $\mathbb{C}^*$ is a necessary condition. It is important to emphasize that our definition only requires $\mathbb{R}^*$-invariance, which is weaker than $\mathbb{C}^*$-invariance. Our definition corresponds to the *absolutely constructible subset* as defined in [WB20, Definition 6.3.1], with the additional condition that $\mathfrak{C}(\mathbb{C})$ is preserved by the action of $\mathbb{R}^*$.

By [WB20, Theorem 9.1.2.(2) & Proposition 7.4.4.(2)] we have the following result, which generalizes [Sim93].

Theorem 1.19 **(Budur-Wang, Simpson).** — *Let X be a smooth projective variety over $\mathbb{C}$. If $\mathfrak{C} \subset M_{\mathrm{B}}(X, 1)(\mathbb{C})$ is an absolute constructible subset, then $\mathfrak{C} = \cup_{i=1}^m N_i^\circ$ where each N_i° is a Zariski dense open subset of a torsion-translated subtori N_i of $M_{\mathrm{B}}(X, 1)$. Moreover, let A be the Albanese variety of X. Then there are abelian subvarieties $P_i \subset A$ such that N_i is the torsion translate of the image in $M_{\mathrm{B}}^0(X, 1) \simeq M_{\mathrm{B}}(A, 1)$ of $M_{\mathrm{B}}(A/P_i, 1)$. Here $M_{\mathrm{B}}^0(X, 1)$ denotes the connected component of $M_{\mathrm{B}}^0(X, 1)$ containing the identity.* □

Absolute constructible subsets are preserved by the following operations:

Theorem 1.20 **(Simpson).** — *Let $f : Z \to X$ be a morphism between smooth projective varieties over $\mathbb{C}$ and let $g : G \to G'$ be a morphism of reductive groups over $\bar{\mathbb{Q}}$. Consider the natural map $i : M_{\mathrm{B}}(X, G) \to M_{\mathrm{B}}(X, G')$ and $j : M_{\mathrm{B}}(X, G) \to M_{\mathrm{B}}(Z, G)$. Then for any absolutely constructible subsets $\mathfrak{C} \subset M_{\mathrm{B}}(X, G)(\mathbb{C})$ and $\mathfrak{C}' \subset M_{\mathrm{B}}(X, G')(\mathbb{C})$, we have $i(\mathfrak{C})$, $i^{-1}(\mathfrak{C}')$ and $j(\mathfrak{C})$ are all absolutely constructible.* □

Example 1.21. — $M_{\mathrm{B}}(X, G)(\mathbb{C})$, the isolated point in $M_{\mathrm{B}}(X, G)(\mathbb{C})$, and the class of trivial representation in $M_{\mathrm{B}}(X, G)(\mathbb{C})$ are all absolutely constructible.

In this paper, absolutely constructible subsets are used to prove the holomorphic convexity of some topological Galois covering of X in Theorems B and C. It will not be used in the proof of Theorem A.

1.9. Katzarkov-Eyssidieux reduction and canonical currents. — For this subsection, we refer to the papers [Eys04, §3.3.2] or [CDY22] for a comprehensive and systematic treatment.

Theorem 1.22 **(Katzarkov, Eyssidieux).** — *Let X be a projective normal variety, and let K be a non-archimedean local field. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation. Then there exists a fibration $s_\varrho : X \to S_\varrho$ to a normal projective space, such that for any subvariety Z of X, the image $\varrho(\mathrm{Im}\,[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is bounded if and only if $s_\varrho(Z)$ is a point. Moreover, if X is smooth, then the above property holds for $\varrho(\mathrm{Im}\,[\pi_1(Z) \to \pi_1(X)])$ without requiring the normalization of Z.*

We will call the above s_ϱ the (Katzarkov-Eyssidieux) reduction map for ϱ. When X is smooth this theorem is proved by Katzarkov [Kat97] and Eyssidieux [Eys04]. It is easier to derive the singular case from their theorem.

Proof of Theorem 1.22. — Let $\mu : Y \to X$ be a resolution of singularities. Since X is normal, $\mu_* : \pi_1(Y) \to \pi_1(X)$ is surjective and thus $\mu^*\varrho : \pi_1(Y) \to \mathrm{GL}_N(K)$ is reductive. By the original theorem of Katzarkov-Eyssidieux, there exists a surjective proper fibration $s_{\mu^*\varrho} : Y \to S_{\mu^*\varrho}$ such that, for any closed subvariety $Z \subset Y$, $s_{\mu^*\varrho}(Z)$ is a point if and only if $\mu^*\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(Y)])$.

If Z is an irreducible component of a fiber of μ. Note that $\mu^*\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(Y)]) = \{1\}$, it follows that $s_{\mu^*\varrho}(Z)$ is a point by the proper of Katzarkov-Eyssidieux. Since each fiber of μ is connected, $s_{\mu^*\varrho}$ contracts each fiber of μ to a point, and it thus descends to a morphism $s_\varrho : X \to S_{\mu^*\varrho}$ such that $s_{\mu^*\varrho} = s_\varrho \circ \mu$.

Let $W \subset X$ be any closed subvariety. Then there exist a closed subvariety $Z \subset Y$ such that $\mu(Z) = W$. Note that $\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(W^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(W^{\mathrm{norm}})$. Therefore, $s_\varrho(Z)$ is a point if and only if $s_{\mu^*\varrho}(W)$ is a point. This condition is equivalent to $\mu^*\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(Y)]) = \varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ being bounded. In turn, this is equivalent to $\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X)])$ being bounded since $\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(W^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(W^{\mathrm{norm}})$. □

We will outline the construction of certain *canonical* positive closed $(1,1)$-currents over S_ϱ. As demonstrated in the proof of [Eys04], we can establish the existence of a finite ramified Galois cover denoted by $\pi : X^{\mathrm{sp}} \to X$ with the Galois group H, commonly known as the *spectral covering of* X (cf. [CDY22, Definition 5.14]). This cover possesses holomorphic 1-forms $\eta_1, \ldots, \eta_m \subset H^0(X^{\mathrm{sp}}, \pi^*\Omega^1_X)$, which can be considered as the $(1,0)$-part of the complexified differential of the $\pi^*\varrho$-equivariant harmonic mapping from X^{sp} to the Bruhat-Tits building of G. These particular 1-forms, referred to as the *spectral one-forms* (cf. [CDY22, Definition 5.16]), play a significant role in the proof of Theorems B and C. Consequently, the Stein factorization of the *partial Albanese morphism* $a : X^{\mathrm{sp}} \to A$ (cf. [CDY22, Definition 5.19]) induced by $\eta_1, \ldots, \eta_m$ leads to the Katzarkov-Eyssidieux reduction map $s_{\pi^*\varrho} : X^{\mathrm{sp}} \to S_{\pi^*\varrho}$ for $\pi^*\varrho$. Moreover, we have the following commutative diagram:

$$\begin{tikzcd} X^{\mathrm{sp}} \arrow[r, "\pi"] \arrow[d, "s_{\pi^*\varrho}"] \arrow[dd, bend right, "a"'] & X \arrow[d, "s_\varrho"] \\ S_{\pi^*\varrho} \arrow[r, "\sigma_\pi"] \arrow[d, "b"] & S_\varrho \\ A & \end{tikzcd}$$

Here σ_π is also a finite ramified Galois cover with Galois group H. Note that there are one forms $\{\eta'_1, \ldots, \eta'_m\} \subset H^0(A, \Omega^1_A)$ such that $a^*\eta'_i = \eta_i$. Consider the finite morphism $b : S_{\pi^*\varrho} \to A$. Then we define a positive $(1,1)$-current $T_{\pi^*\varrho} := b^* \sum_{i=1}^m i\eta'_i \wedge \overline{\eta_i}'$ on $S_{\pi^*\varrho}$. Note that $T_{\pi^*\varrho}$ is invariant under the Galois action H. Therefore, by Lemma 1.10 there is a positive closed $(1,1)$-current T_ϱ defined on S_ϱ with continuous potential such that $\sigma_\pi^* T_\varrho = T_{\pi^*\varrho}$.

***Definition 1.23* (Canonical current).** — The closed positive $(1,1)$-current T_ϱ on S_ϱ is called the *canonical current* of ϱ.

More generally, let $\{\varrho_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,k}$ be reductive representations where K_i is a non-archimedean local field. We shall denote by the bolded letter $\boldsymbol{\varrho} := \{\varrho_i\}_{i=1,\ldots,k}$ be such family of representations. Let $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ be the Stein factorization of $(s_{\varrho_1}, \ldots, s_{\varrho_k}) : X \to S_{\varrho_1} \times \cdots \times S_{\varrho_k}$ where $s_{\varrho_i} : X \to S_{\varrho_i}$ denotes the reduction map associated with ϱ_i and $p_i : S_{\boldsymbol{\varrho}} \to S_{\varrho_i}$ is the induced finite morphism. $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ is called the *reduction map* for the family $\boldsymbol{\varrho}$ of representations.

***Definition 1.24* (Canonical current II).** — The closed positive $(1,1)$-current $T_{\boldsymbol{\varrho}} := \sum_{i=1}^k p_i^* T_{\varrho_i}$ on $S_{\boldsymbol{\varrho}}$ is called the canonical current of $\boldsymbol{\varrho}$.

***Lemma 1.25* ([Eys04, Lemme 1.4.9 & 3.3.10]).** — *Let $f : Z \to X$ be a morphism between projective normal varieties and let $\boldsymbol{\varrho} := \{\varrho_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,k}$ be a family of reductive representations where K_i is a non-archimedean local field. Then we* *have*

$$
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "s_{f^*\varrho}"'] & X \arrow[d, "s_{\varrho}"] \\
S_{f^*\varrho} \arrow[r, "\sigma_f"] & S_{\varrho}
\end{tikzcd}
\tag{1.2}
$$

where σ_f is a finite *morphism. Here $f^*\varrho = \{f^*\varrho_i\}_{i=1,\ldots,k}$ denotes the pull back of the family of $\varrho := \{\varrho_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,k}$. Moreover, the following properties hold:*

(i) *The local potential of T_{ϱ} is continuous. In particular, for any closed subvariety $W \subset X$, we have*
$$\{T_{\varrho}\}^{\dim W} \cdot W = \int_W T_{\varrho}^{\dim W} \geq 0.$$
(ii) $T_{f^*\varrho} = \sigma_f^* T_{\varrho}$;
(iii) *For every closed subvariety $\Xi \subset S_{f^*\varrho}$, $\{T_{\varrho}\}^{\dim \Xi} \cdot (\sigma_f(\Xi)) > 0$ if and only if $\{T_{f^*\varrho}\}^{\dim \Xi} \cdot \Xi > 0$.* □

Note that Lemma 1.25.(iii) is a consequence of the first two assertions.

The current T_ϱ will serve as a lower bound for the complex hessian of plurisubharmonic functions constructed by the method of harmonic mappings.

***Proposition 1.26* ([Eys04, Proposition 3.3.6, Lemme 3.3.12])** *Let X be a projective normal variety and let $\varrho : \pi_1(X) \to G(K)$ be a Zariski dense representation where K is a non archimedean local field and G is a reductive group. Let $x_0 \in \Delta(G)$ be an arbitrary point. Let $u : \widetilde{X} \to \Delta(G)$ be the associated the harmonic mapping, where $\widetilde{X}$ is the universal covering of X. The function $\phi : \widetilde{X} \to \mathbb{R}_{\geq 0}$ defined by*
$$\phi(x) = 2d^2\left(u(x), u(x_0)\right)$$
satisfies the following properties:

(a) *ϕ descends to a function ϕ_ϱ on $\widetilde{X_\varrho} = \widetilde{X}/\ker(\varrho)$.*
(b) $\mathrm{dd^c}\phi_\varrho \geq (s_\varrho \circ \pi)^* T_\varrho$, *where we denote by $\pi : \widetilde{X}_\varrho \to X$ the covering map.*
(c) *ϕ_ϱ is locally Lipschitz;*
(d) *Let T be a normal complex space and $r : \widetilde{X}_\varrho \to T$ a proper holomorphic fibration such that $s_\varrho \circ \pi : \widetilde{X}_\varrho \to S_\varrho$ factorizes via a morphism $\nu : T \to S_\varrho$. The function ϕ_ϱ is of the form $\phi_\varrho = \phi_\varrho^T \circ r$ with ϕ_ϱ^T being a continuous plurisubharmonic function on T;*
(e) $\mathrm{dd^c}\phi_\varrho^T \geq \nu^* T_\varrho$. □

1.10. The generalization of Katzarkov-Eyssidieux reduction to quasi-projective varieties. — In our work [CDY22] on hyperbolicity of quasi-projective varieties, we extended Theorem 1.22 to quasi-projective varieties. The theorem we established is stated below.

***Theorem 1.27* ([CDY22, Theorem 0.10]).** — *Let X be a complex smooth quasi-projective variety, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a reductive representation where K is non-archimedean local field. Then there exists a quasi-projective normal variety S_ϱ and a dominant morphism $s_\varrho : X \to S_\varrho$ with connected general fibers, such that for any connected Zariski closed subset T of X, the following properties are equivalent:*

(a) *the image $\varrho(\mathrm{Im}[\pi_1(T) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.*
(b) *For every irreducible component T_o of T, the image $\varrho(\mathrm{Im}[\pi_1(T_o^{\mathrm{norm}}) \to \pi_1(X)])$ is a bounded subgroup of $G(K)$.*
(c) *The image $s_\varrho(T)$ is a point.* □

This result plays a crucial role in the proof of Theorem A. Its proof is built upon the work by Brotbek, Daskalopoulos, Mese, and the first named author [BDDM22], regarding the construction of ϱ-equivariant harmonic mappings from the universal covering of X to the Bruhat-Tits building $\Delta(G)$ of G.

1.11. Simultaneous Sten factorization. —

Lemma 1.28. — *Let V be a smooth quasi-projective variety. For $i = 1, 2, \ldots$, let W_i be normal quasi-projective varieties such that*

— *there exist dominant morphisms $p_i : V \to W_i$, and*
— *there exist dominant morphisms $q_i : W_i \to W_{i-1}$ such that $q_i \circ p_i = p_{i-1}$.*

Then there exists $i_0 \in \mathbb{Z}_{\geq 2}$ such that for every $i \geq i_0$ and every subvariety $Z \subset V$, if $p_{i-1}(Z)$ is a point, then $p_i(Z)$ is a point.

Proof. — Let $E_i \subset V \times V$ be defined by

$$E_i = \{(x, x') \in V \times V; p_i(x) = p_i(x')\}.$$

Then $E_i \subset V \times V$ is a Zariski closed set. Indeed, $E_i = (p_i, p_i)^{-1}(\Delta_i)$, where $(p_i, p_i) : V \times V \to W_i \times W_i$ is the morphism defined by $(p_i, p_i)(x, x') = (p_i(x), p_i(x'))$ and $\Delta_i \subset W_i \times W_i$ is the diagonal. Now by $q_i \circ p_i = p_{i-1}$, we have $E_i \subset E_{i-1}$. By the Noetherian property, there exists i_0 such that $E_{i+1} = E_i$ for all $i \geq i_0$. Then the induced map $p_{i+1}(V) \to p_i(V)$ is injective. Hence if $p_{i-1}(Z)$ is a point, then $p_i(Z)$ is a point. □

Lemma 1.29. — *Let V be a quasi-projective normal variety and let $(f_\lambda : V \to S_\lambda)_{\lambda \in \Lambda}$ be a family of morphisms into quasi-projective varieties S_λ. Then there exist a normal projective variety S_∞ and a morphism $f_\infty : V \to S_\infty$ such that*

— *for every subvariety $Z \subset V$, if $f_\infty(Z)$ is a point, then $f_\lambda(Z)$ is a point for every $\lambda \in \Lambda$, and*
— *there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $f_\infty : V \to S_\infty$ is the quasi-Stein factorization of $(f_1, \ldots, f_n) : V \to S_{\lambda_1} \times \cdots S_{\lambda_n}$.*

Proof. — We take $\lambda_1 \in \Lambda$. Let $p_1 : V \to W_1$ be the quasi-Stein factorization of $f_{\lambda_1} : V \to S_{\lambda_1}$.

Next we take (if it exists) $\lambda_2 \in \Lambda$ such that for the quasi-Stein factorization $p_2 : V \to W_2$ of $(s_{\lambda_1}, s_{\lambda_2}) : X \to S_{\lambda_1} \times S_{\lambda_2}$, there exists a subvariety $Z \subset V$ such that $p_1(Z)$ is a point, but $p_2(Z)$ is not a point.

Similarly, we take (if it exists) $\lambda_3 \in \Lambda$ such that for the Stein factorization $p_3 : V \to W_3$ of $(f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}) : V \to S_{\lambda_1} \times S_{\lambda_2} \times S_{\lambda_3}$, there exists a subvariety $Z \subset V$ such that $p_2(Z)$ is a point, but $p_3(Z)$ is not a point.

We repeat this process forever we may continue. However by Lemma 1.28, this process should terminate to get $\lambda_1, \ldots, \lambda_n \in \Lambda$. We let $S_\infty = W_n$, namely $f_\infty : V \to S_\infty$ is the Stein factorization of $(f_{\lambda_1}, \ldots, f_{\lambda_n}) : V \to S_{\lambda_1} \times \cdots \times S_{\lambda_n}$.

Now let $\lambda \in \Lambda$. Then by the construction, if $f_\infty(Z)$ is a point, then $(f_{\lambda_1}, \ldots, f_{\lambda_n}, f_\lambda)(Z)$ is a point. In particular, $f_\lambda(Z)$ is a point. □

We also need the following generalized Stein factorization proved by Henri Cartan in [Car60, Theorem 3].

Theorem 1.30. — *Let X, S be complex spaces and $f : X \to S$ be a morphism. Suppose a connected component F of a fibre of f is compact. Then, F has an open neighborood V such that $f(V)$ is a locally closed analytic subvariety S and $V \to f(V)$ is proper.*

Suppose furthermore that X is normal and that every connected component F of a fibre of f is compact. The set Y of connected components of fibres of f can be endowed with the structure of a normal complex space such that f factors through the natural map $e : X \to Y$ which is a proper holomorphic fibration. □

2. Some non-abelian Hodge theories

In this section, we will build upon the previous work of Simpson [Sim92], Iyer-Simpson [IS07, IS08], and Mochizuki [Moc07a, Moc06] to further develop non-abelian Hodge theories over quasi-projective varieties. We begin by establishing the functoriality of pullback for regular filtered Higgs bundles (cf. Proposition 2.5). Then we clarify the $\mathbb{C}^*$ and $\mathbb{R}^*$-action on the character varieties of smooth quasi-projective varieties, following [Moc06]. Lastly, we prove Proposition 2.9, which essentially states that the natural morphisms of character varieties induced by algebraic morphisms commute with the $\mathbb{C}^*$-action. This section's significance lies in its essential role in establishing Propositions 3.12 and 3.35, which serves as a critical cornerstone of the whole paper.

2.1. Regular filtered Higgs bundles. — In this subsection, we recall the notions of regular filtered Higgs bundles (or parabolic Higgs bundles). For more details refer to [Moc06]. Let $\overline{X}$ be a complex manifold with a reduced simple normal crossing divisor $D=\sum_{i=1}^{\ell}D_i$, and let $X=\overline{X}\backslash D$ be the complement of D. We denote the inclusion map of X into $\overline{X}$ by j.

Definition 2.1. — A *regular filtered Higgs bundle* $(\boldsymbol{E}_*,\theta)$ on $(\overline{X},D)$ is holomorphic vector bundle E on X, together with an $\mathbb{R}^\ell$-indexed filtration ${}_{\boldsymbol{a}}E$ (so-called *parabolic structure*) by locally free subsheaves of j_*E such that

1. $\boldsymbol{a}\in\mathbb{R}^\ell$ and ${}_{\boldsymbol{a}}E|_X=E$.
2. For $1\leq i\leq \ell$, ${}_{\boldsymbol{a}+\boldsymbol{1}_i}E={}_{\boldsymbol{a}}E\otimes\mathcal{O}_X(D_i)$, where $\boldsymbol{1}_i=(0,\ldots,0,1,0,\ldots,0)$ with 1 in the i-th component.
3. ${}_{\boldsymbol{a}+\boldsymbol{\epsilon}}E={}_{\boldsymbol{a}}E$ for any vector $\boldsymbol{\epsilon}=(\epsilon,\ldots,\epsilon)$ with $0<\epsilon\ll 1$.
4. The set of *weights* $\{\boldsymbol{a}\mid {}_{\boldsymbol{a}}E/{}_{\boldsymbol{a}-\boldsymbol{\epsilon}}E\neq 0$ for any vector $\boldsymbol{\epsilon}=(\epsilon,\ldots,\epsilon)$ with $0<\epsilon\ll 1\}$ is discrete in $\mathbb{R}^\ell$.
5. There is a $\mathcal{O}_{\overline{X}}$-linear map, so-called Higgs field,
$$\theta:{}^\diamond E\to\Omega^1_{\overline{X}}(\log D)\otimes{}^\diamond E$$
such that $\theta\wedge\theta=0$, and

$$\theta({}_{\boldsymbol{a}}E)\subseteq\Omega^1_{\overline{X}}(\log D)\otimes{}_{\boldsymbol{a}}E. \tag{2.1}$$

Denote ${}_{\boldsymbol{0}}E$ by ${}^\diamond E$, where $\boldsymbol{0}=(0,\ldots,0)$. When disregarding the Higgs field, $\boldsymbol{E}_*$ is referred to as a *parabolic bundle*. By the work of Borne-Vistoli the parabolic structure of a parabolic bundle is *locally abelian*, *i.e.* it admits a local frame compatible with the filtration (see e.g. [IS07]).

A natural class of regular filtered Higgs bundles comes from prolongations of tame harmonic bundles. We first recall some notions in [Moc07a, §2.2.1]. Let E be a holomorphic vector bundle with a smooth hermitian metric h over X.

Let U be an open subset of $\overline{X}$ with an admissible coordinate $(U;z_1,\ldots,z_n)$ with respect to D. For any section $\sigma\in\Gamma(U\backslash D,E|_{U\backslash D})$, let $|\sigma|_h$ denote the norm function of σ with respect to the metric h. We denote $|\sigma|_h\mathcal{O}(\prod_{i=1}^{\ell}|z_i|^{-b_i})$ if there exists a positive number C such that $|\sigma|_h\leq C\cdot\prod_{i=1}^{\ell}|z_i|^{-b_i}$. For any $\boldsymbol{b}\in\mathbb{R}^\ell$, say $-\mathrm{ord}(\sigma)\leq\boldsymbol{b}$ means the following:
$$|\sigma|_h=\mathcal{O}(\prod_{i=1}^{\ell}|z_i|^{-b_i-\varepsilon})$$
for any real number $\varepsilon>0$ and $0<|z_i|\ll 1$. For any $\boldsymbol{b}$, the sheaf ${}_{\boldsymbol{b}}E$ is defined as follows:
$$\Gamma(U,{}_{\boldsymbol{b}}E):=\{\sigma\in\Gamma(U\backslash D,E|_{U\backslash D})\mid -\mathrm{ord}(\sigma)\leq\boldsymbol{b}\}. \tag{2.2}$$
The sheaf ${}_{\boldsymbol{b}}E$ is called the prolongment of E by an increasing order $\boldsymbol{b}$. In particular, we use the notation ${}^\diamond E$ in the case $\boldsymbol{b}=(0,\ldots,0)$.

According to Simpson [Sim90, Theorem 2] and Mochizuki [Moc07a, Theorem 8.58], the above prolongation gives a regular filtered Higgs bundle.

***Theorem 2.2* (Simpson, Mochizuki).** — *Let $\overline{X}$ be a complex manifold and D be a simple normal crossing divisor on $\overline{X}$. If (E,θ,h) is a tame harmonic bundle on $\overline{X}\backslash D$, then the corresponding filtration ${}_{\boldsymbol{b}}E$ defined above defines a regular filtered Higgs bundle $(\boldsymbol{E}_*,\theta)$ on $(\overline{X},D)$.* □

2.2. Pullback of parabolic bundles. — In this subsection, we introduce the concept of pullback of parabolic bundles. We refer the readers to [IS07, IS08] for a more systematic treatment. We avoid the language of Deligne-Mumford stacks in [IS07, IS08]. This subsection is conceptional and we shall make precise computations in next subsection.

A parabolic line bundle is a parabolic sheaf F such that all the ${}_{\boldsymbol{a}}F$ are line bundles. An important class of examples is obtained as follows: let L be a line bundle on $\overline{X}$, if $\boldsymbol{a} = (a_1, \ldots, a_\ell)$ is a $\mathbb{R}^\ell$-indexed, then we can define a parabolic line bundle denoted $L_*^{\boldsymbol{a}}$ by setting

$$ {}_{\boldsymbol{b}}L^{\boldsymbol{a}} := L \otimes \mathcal{O}_{\overline{X}}\left(\sum_{i=1}^{\ell} \lfloor a_i + b_i \rfloor D_i\right) \tag{2.3}$$

for any $\boldsymbol{b} \in \mathbb{R}^\ell$.

***Definition 2.3* (Locally abelian parabolic bundle).** — A parabolic sheaf $\boldsymbol{E}_*$ is a *locally abelian parabolic bundle* if, in a neighborhood of any point $x \in \overline{X}$ there is an isomorphism between $\boldsymbol{E}_*$ and a direct sum of parabolic line bundles.

Let $f : \overline{Y} \to \overline{X}$ be a holomorphic map of complex manifolds. Let $D' = \sum_{j=1}^{k} D'_j$ and $D = \sum_{i=1}^{\ell} D_i$ be simple normal crossing divisors on $\overline{Y}$ and $\overline{X}$ respectively. Assume tht $f^{-1}(D) \subset D'$. Denote by $n_{ij} = \mathrm{ord}_{D'_j} f^* D_i \in \mathbb{Z}_{\geq 0}$. Let L be a line bundle on $\overline{X}$ and let $L_*^{\boldsymbol{a}}$ be the parabolic line bundle defined in (2.3). Set

$$ f^*\boldsymbol{a} := (\sum_{i=1}^{\ell} n_{i1} a_i, \ldots, \sum_{i=1}^{\ell} n_{ik} a_i) \in \mathbb{R}^k. \tag{2.4}$$

Then $f^*(L_*^{\boldsymbol{a}})$ is defined by setting

$$ {}_{\boldsymbol{b}}(f^*L)^{f^*\boldsymbol{a}} := f^*L \otimes \mathcal{O}_{\overline{Y}}\left(\sum_{j=1}^{k} \lfloor \sum_{i=1}^{\ell} n_{ij} a_i + b_j \rfloor D'_j\right) \tag{2.5}$$

for any $\boldsymbol{b} \in \mathbb{R}^k$.

Let $\overline{X}$ be a compact complex manifold. Consider a locally abelian parabolic bundle $\boldsymbol{E}_*$ defined on $\overline{X}$. We can cover $\overline{X}$ with open subsets $U_1, \ldots, U_m$, such that $\boldsymbol{E}_*|_{U_i}$ can be expressed as a direct sum of parabolic line bundles on each U_i.

Using this decomposition, we define the pullback $f^*(\boldsymbol{E}_*|_{U_i})$ as in (2.5). It can be verified that $f^*(\boldsymbol{E}_*|_{U_i})$ is compatible with $f^*(\boldsymbol{E}_*|_{U_j})$ whenever $U_i \cap U_j \neq \varnothing$. This allows us to extend the local pullback to a global level, resulting in the definition of the pullback of a locally abelian parabolic bundle denoted by $f^*\boldsymbol{E}_*$. In next section, we will see an explicit description of the pullback of regular filtered Higgs bundles induced by tame harmonic bundles.

2.3. Functoriality of pullback of regular filtered Higgs bundle. — We recall some notions in [Moc07a, §2.2.2]. Let X be a complex manifold, D be a simple normal crossing divisor on X, and E be a holomorphic vector bundle on $X \backslash D$ such that $E|_{X \backslash D}$ is equipped with a hermitian metric h. Let $\boldsymbol{v} = (v_1, \ldots, v_r)$ be a smooth frame of $E|_{X \backslash D}$. We obtain the $H(r)$-valued function $H(h, \boldsymbol{v})$ defined over $X \backslash D$,whose (i, j)-component is given by $h(v_i, v_j)$.

Let us consider the case $X = \mathbb{D}^n$, and $D = \sum_{i=1}^{\ell} D_i$ with $D_i = (z_i = 0)$. We have the coordinate $(z_1, \ldots, z_n)$. Let h, E and $\boldsymbol{v}$ be as above.

Definition 2.4. — A smooth frame $\boldsymbol{v}$ on $X \backslash D$ is called *adapted up to log order*, if the following inequalities hold over $X \backslash D$:

$$ C^{-1}(-\sum_{i=1}^{\ell} \log |z_i|)^{-M} \leq H(h, \boldsymbol{v}) \leq C(-\sum_{i=1}^{\ell} \log |z_i|)^{M} $$

for some positive numbers M and C.

The goal of this subsection is to establish the following result concerning the functoriality of the pullback of a regular filtered Higgs bundle. This result will play a crucial role in proving Proposition 3.35.

Proposition 2.5. — *Consider a morphism $f:\overline{Y}\to\overline{X}$ of smooth projective varieties $\overline{X}$ and $\overline{Y}$. Let D and D' be simple normal crossing divisors on $\overline{X}$ and $\overline{Y}$ respectively. Assume that $f^{-1}(D)\subset D'$. Let (E,θ,h) be a tame harmonic bundle on $X:=\overline{X}\backslash D$. Let $(\boldsymbol{E}_*,\theta)$ be the regular filtered Higgs bundle defined in § 2.1. Consider the pullback of $f^*\boldsymbol{E}_*$ defined in § 2.2, which is also a parabolic bundle over $(\overline{Y},D')$. Then*

(i) *$f^*\boldsymbol{E}_*$ is the prolongation $\tilde{E}_*$ of f^*E using the norm growth with respect to the metric f^*h as defined in (2.2).*
(ii) *$(f^*\boldsymbol{E}_*,f^*\theta)$ is a filtered regular Higgs bundle.*

Proof. — Since this is a local result, we assume that $\overline{X}:=\mathbb{D}^n$ and $D:=\bigcup_{i=1}^{\ell}\{z_i=0\}$. Let $\overline{Y}:=\mathbb{D}^m$ and $D':=\bigcup_{j=1}^{k}\{w_j=0\}$. Then, $f^*(z_i)=\prod_{j=1}^{k}w_j^{n_{ij}}g_i$ for some invertible functions $\{g_i\}_{i=1,\ldots,\ell}\subset\mathcal{O}(\overline{Y})$.

By [Moc07a, Proposition 8.70], there exists a holomorphic frame $\boldsymbol{v}=(v_1,\ldots,v_r)$ of ${}^\diamond E|_{\overline{X}}$ and $\{a_{ij}\}_{i=1,\ldots,r;j=1,\ldots,\ell}\subset\mathbb{R}$ such that if we put $\tilde{v}_i:=v_i\cdot\prod_{j=1}^{\ell}|z_j|^{-a_{ij}}$, then for the smooth frame $\widetilde{\boldsymbol{v}}=(\tilde{v}_1,\ldots,\tilde{v}_r)$ over $X=\overline{X}\backslash D$, $H(h,\widetilde{\boldsymbol{v}})$ is adapted to log order in the sense of Definition 2.4.

Define L_i to be the sub-line bundle of ${}^\diamond E$ generated by v_i. Write $\boldsymbol{a}_i:=(a_{i1},\ldots,a_{i\ell})\in\mathbb{R}^\ell$. Consider the parabolic line bundle $(L_i)_*^{\boldsymbol{a}_i}$ over $(\overline{X},D)$ defined in (2.3), namely,

$$
{}_{\boldsymbol{b}}(L_i)^{\boldsymbol{a}_i}:=L_i\otimes\mathcal{O}_{\overline{X}}\left(\sum_{j=1}^{\ell}\lfloor a_{ij}+b_j\rfloor D_j\right)\tag{2.6}
$$

for any $\boldsymbol{b}\in\mathbb{R}^\ell$.

Claim 2.6. — *The parabolic bundles $\boldsymbol{E}_*$ and $\oplus_{i=1}^{r}(L_i)_*^{\boldsymbol{a}_i}$ are the same. In particular, $\boldsymbol{E}_*$ is locally abelian.*

Proof. — By (2.2), for any $\boldsymbol{b}\in\mathbb{R}^\ell$, any holomorphic section $\sigma\in\Gamma(\overline{X},{}_{\boldsymbol{b}}E)$ satisfies

$$
|\sigma|_h=\mathcal{O}(\prod_{j=1}^{\ell}|z_j|^{-b_j-\varepsilon}).
$$

As $\boldsymbol{v}$ is a frame for ${}^\diamond E$, one can write $\sigma=\sum_{i=1}^{r}g_iv_i$ where g_i is a holomorphic function defined on X. Write $\boldsymbol{g}:=(g_1,\ldots,g_r)$. Since $H(h,\widetilde{\boldsymbol{v}})$ is adapted to log order, it follows that

$$
C^{-1}(-\sum_{j=1}^{\ell}\log|z_j|)^{-M}\cdot\sum_{i=1}^{r}|g_i|^2\prod_{j=1}^{\ell}|z_j|^{2a_{ij}}\leq\overline{\boldsymbol{g}}H(h,\boldsymbol{v})\boldsymbol{g}^T=|\sigma|_h^2=\mathcal{O}(\prod_{j=1}^{\ell}|z_j|^{-2b_i-\varepsilon})
$$

for any $\varepsilon>0$. Hence for each i and any $\varepsilon>0$ we have

$$
|g_i|^2=\mathcal{O}(\prod_{j=1}^{\ell}|z_j|^{-2(b_j+a_{ij})-\varepsilon}).
$$

Therefore, $\operatorname{ord}_{D_j}g_i\leq-\lfloor b_j+a_{ij}\rfloor$. This proves that

$$
{}_{\boldsymbol{b}}E\subset\oplus_{i=1}^{r}{}_{\boldsymbol{b}}(L_i)^{\boldsymbol{a}_i}.
$$

On the other hand, we consider any section $\sigma\in\Gamma(\overline{X},{}_{\boldsymbol{b}}(L_i)^{\boldsymbol{a}_i})$. Then $\sigma=gv_i$ for some meromorphic function g defined over $\overline{X}$ such that $\operatorname{ord}_{D_j}g_i\leq-\lfloor b_j+a_{ij}\rfloor$ by (2.6). Therefore, there exists some positive constant $C>0$ such that

$$
|\sigma|_h^2=|g|^2|v_i|_h^2\leq C\prod_{j=1}^{\ell}|z_j|^{-2(b_j+a_{ij})}\cdot|\tilde{v}_i|_h^2\cdot\prod_{j=1}^{\ell}|z_j|^{2a_{ij}}=C\prod_{j=1}^{\ell}|z_i|^{-2b_j}\cdot|\tilde{v}_i|_h^2=\mathcal{O}(\prod_{i=1}^{\ell}|z_i|^{-b_i-\varepsilon}).
$$

as $|\tilde{v}_i|_h^2 \leq C(-\sum_{j=1}^{\ell} \log|z_j|)^M$ for some $C, M > 0$. This implies that

$$\oplus_{i=1}^r {}_{\boldsymbol{b}}(L_i)^{\boldsymbol{a}_i} \subset {}_{\boldsymbol{b}}E.$$

The claim is proved. □

Consider the pullback $f^*\boldsymbol{v} := (f^*v_1, \ldots, f^*v_m)$. Then it is a holomorphic frame of $f^*E|_Y$ where $Y := \overline{Y} \backslash D'$. Note that we have

$$f^*\tilde{v}_i := f^*v_i \cdot \prod_{j=1}^{\ell} |f^*z_j|^{-a_{ij}} = f^*v_i \cdot \prod_{j=1}^{\ell} \prod_{q=1}^{k} |w_q|^{-n_{jq}a_{ij}} \cdot g_i'$$

for some invertible holomorphic function $g_i' \in \mathcal{O}(\overline{Y})$. Similar to (2.4), we set

$$f^*\boldsymbol{a}_i := (\sum_{j=1}^{\ell} n_{j1}a_{ij}, \ldots, \sum_{j=1}^{\ell} n_{jk}a_{ij}) \in \mathbb{R}^k.$$

Then we have

$$f^*\tilde{v}_i := f^*v_i \cdot |\boldsymbol{w}^{-f^*\boldsymbol{a}_i}| \cdot g_i'.$$

Since $H(h, \widetilde{\boldsymbol{v}})$ is adapted to log order, it is easy to check that $H(f^*h, f^*\widetilde{\boldsymbol{v}})$ also is adapted to log order. Set $e_i := f^*v_i \cdot |\boldsymbol{w}^{-f^*\boldsymbol{a}_i}|$ for $i = 1, \ldots, r$ and $\boldsymbol{e} := (e_1, \ldots, e_r)$. Then $\boldsymbol{e}$ is a smooth frame for $f^*E|_Y$. Since g_i' is invertible, it follows that $H(f^*h, \boldsymbol{e})$ is also adapted to log order. Consider the prolongation $(\tilde{E}_*, \tilde{\theta})$ of the tame harmonic bundle $(f^*E, f^*\theta, f^*h)$ using the norm growth as defined in (2.2). Applying the result from Claim 2.6 to $(f^*E, f^*\theta, f^*h)$, we can conclude that the parabolic bundle $\tilde{E}_*$ is given by

$$\tilde{E}_* = \oplus_{i=1}^r (f^*L_i)_*^{f^*\boldsymbol{a}_i}, \tag{2.7}$$

where $(f^*L_i)_*^{f^*\boldsymbol{a}_i}$ are parabolic line bundles defined by

$${}_{\boldsymbol{b}}(f^*L_i)^{f^*\boldsymbol{a}_i} := f^*L_i \otimes \mathcal{O}_{\overline{Y}}\left(\sum_{j=1}^{k} \lfloor \sum_{q=1}^{\ell} n_{qj}a_{iq} + b_j \rfloor D_j'\right). \tag{2.8}$$

On the other hand, by our definition of pullback of parabolic bundles and Claim 2.6, we have

$$f^*\boldsymbol{E}_* := \oplus_{i=1}^r f^*(L_i)_*^{\boldsymbol{a}_i}$$

where $f^*(L_i)_*^{\boldsymbol{a}_i}$ is the pullback of parabolic line bundle $(L_i)_*^{\boldsymbol{a}_i}$ defined in (2.5). By performing a straightforward computation, we find that

$$f^*(L_i)_*^{\boldsymbol{a}_i} = f^*L_i \otimes \mathcal{O}_{\overline{Y}}\left(\sum_{j=1}^{\ell} \lfloor \sum_{q=1}^{\ell} n_{qj}a_{iq} + b_j \rfloor D_j'\right).$$

This equality together with (2.7) and (2.8) yields $\tilde{E}_* = f^*\boldsymbol{E}_*$. We prove our first assertion. The second assertion can be deduced from the first one, combined with Theorem 2.2. □

2.4. $\mathbb{C}^*$-action and $\mathbb{R}^*$-action on character varieties. — Consider a smooth projective variety $\overline{X}$ equipped with a simple normal crossing divisor D. We define X as the complement of D in $\overline{X}$. Additionally, we fix an ample line bundle L on $\overline{X}$. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation.

According to Theorem 1.6, there exists a tame pure imaginary harmonic bundle (E, θ, h) on X such that $(E, \nabla_h + \theta + \theta_h^\dagger)$ is flat, with the monodromy representation being precisely ϱ. Here ∇_h is the Chern connection of (E, h) and $\theta_h^\dagger$ is the adjoint of θ with respect to h. Let $(\boldsymbol{E}_*, \theta)$ be the prolongation of (E, θ) on $\overline{X}$ defined in § 2.1. By [Moc06, Theorem 1.4], $(\boldsymbol{E}_*, \theta)$ is a μ_L-polystable regular filtered Higgs bundle on $(\overline{X}, D)$ with trivial characteristic numbers. Therefore, for any $t \in \mathbb{C}^*$, $(\boldsymbol{E}_*, t\theta)$ be also μ_L-polystable regular filtered Higgs bundle on $(\overline{X}, D)$ with trivial characteristic numbers. By [Moc06, Theorem 9.4], there is a pluriharmonic metric h_t for $(E, t\theta)$ adapted to the parabolic structures of $(\boldsymbol{E}_*, t\theta)$. Then $(E, t\theta, h_t)$ is a harmonic bundle and thus the connection $\nabla_{h_t} + t\theta + \bar{t}\theta_{h_t}^\dagger$ is flat. Here ∇_{h_t}

is the Chern connection for (E, h_t) and $\theta^\dagger_{h_t}$ is the adjoint ot θ with respect to h_t. Let us denote by $\varrho_t : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ the monodromy representation of $\nabla_{h_t} + t\theta + \bar{t}\theta^\dagger_{h_t}$. It should be noted that the representation ϱ_t is well-defined up to conjugation. As a result, the $\mathbb{C}^*$-action is only well-defined over $M_{\mathrm{B}}(X, N)$ and we shall denote it by

$$t.[\varrho] := [\varrho_t] \quad \text{for any } t \in \mathbb{C}^*.$$

It is important to observe that unlike the compact case, ϱ_t is not necessarily reductive in general, even if the original representation ϱ is reductive. However, if $t \in \mathbb{R}^*$, $(E, t\theta)$ is also pure imaginary and by Theorem 1.6, ϱ_t is reductive. Nonetheless, we can obtain a family of (might not be semisimple) representations $\{\varrho_t : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{t\in\mathbb{C}^*}$. By [Moc06, Proofs of Theorem 10.1 and Lemma 10.2] we have

Lemma 2.7. — *The map*

$$\begin{aligned}\Phi : \mathbb{R}^* &\to M_{\mathrm{B}}(\pi_1(X), N)\\ t &\mapsto [\varrho_t]\end{aligned}$$

is continuous. $\Phi(\{t \in \mathbb{R}^* \mid |t| < 1\})$ *is relatively compact in* $M_{\mathrm{B}}(\pi_1(X), N)$. □

Note that Lemma 2.7 can not be seen directly from [Moc06, Lemma 10.2] as he did not treat the character variety in his paper. Indeed, based on Uhlenbeck's compactness in Gauge theory, Mochizuki's proof can be read as follows: for any $t_n \in \mathbb{R}^*$ converging to 0, after subtracting to a subsequence, there exists some $\varrho_0 : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ and $g_n \in \mathrm{GL}_N(\mathbb{C})$ such that $\lim\limits_{n\to\infty} g_n^*\varrho_{t_n} = \varrho_0$ in the representation variety $R(\pi_1(X), \mathrm{GL}_N)(\mathbb{C})$. Moreover, one can check that ϱ_0 corresponds to some tame pure imaginary harmonic bundle, and thus by Theorem 1.6 it is reductive (cf. [BDDM22] for a more detailed study). For this reason, we can see that it will be more practical to work with $\mathbb{R}^*$-action instead of $\mathbb{C}^*$-action as the representations we encounter are all reductive.

When X is compact, Simpson proved that $\lim_{t\to 0} \Phi(t)$ exists and underlies a $\mathbb{C}$-VHS. However, it is current unknown in the quasi-projective setting. Instead, Mochizuki proved that, we achieve a $\mathbb{C}$-VHS after finite steps of deformations. Let us recall it briefly and the readers can refer to [Moc06, §10.1] for more details.

Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a semisimple representation. Then there exists a tame and pure imaginary harmonic bundle (E, θ, h) corresponding to ϱ. Then the induced regular filtered Higgs bundle $(\boldsymbol{E}_*, \theta)$ on $(\overline{X}, D)$ is μ_L-polystable with trivial characteristic numbers. Hence we have a decomposition

$$(\boldsymbol{E}_*, \theta) = \oplus_{j\in\Lambda}(\boldsymbol{E}_{j*}, \theta_j) \otimes \mathbb{C}^{m_j}$$

where $(\boldsymbol{E}_{j*}, \theta_j)$ is μ_L-stable regular filtered Higgs bundle with trivial characteristic numbers. Put $r(\varrho) := \sum_{j\in\Lambda} m_j$. Then $r(\varrho) \le \operatorname{rank} E$. For any $t \in \mathbb{R}^*$, we know that $(E, t\theta)$ is still tame and pure imaginary and thus ϱ_t is also reductive. Since $\varrho(\{t \in \mathbb{C}^* \mid |t| < 1\})$ is relatively compact, then there exists some $t_n \in \mathbb{R}^*$ which converges to zero such that $\lim_{t_n\to 0}[\varrho_{t_n}]$ exists, denoting by $[\varrho_0]$. Moreover, ϱ_0 corresponds to some tame harmonic bundle. There are two possibilities:

- For each $j \in \Lambda$, $(\boldsymbol{E}_{j*}, t_n\theta_j)$ converges to some μ_L-stable regular filtered Higgs sheaf (cf. [Moc06, p. 96] for the definition of convergence). Then by [Moc06, Proposition 10.3], ϱ_0 underlies a $\mathbb{C}$-VHS.
- For some $j \in \Lambda$, $(\boldsymbol{E}_{j*}, t_n\theta_j)$ converges to some μ_L-semistable regular filtered Higgs sheaf, but not μ_L-stable. Then by [Moc06, Lemma 10.4] $r(\varrho) < r(\varrho_0)$. In other words, letting ϱ_i be the representation corresponding to $(\boldsymbol{E}_{j*}, \theta_j)$ and $\varrho_{i,t}$ be the deformation under $\mathbb{C}^*$-action. Then $\lim_{n\to\infty} \varrho_{i,t_n}$ exists, denoted by $\varrho_{i,0}$. Then $\varrho_{i,0}$ corresponds to some tame harmonic bundle, and thus also a μ_L-polystable regular filtered Higgs bundle which is not stable. In this case, we further deform ϱ_0 until we achieve Case 1.

In summary, Mochizuki's result implies the following, which we shall refer to as *Mochizuki's ubiquity*, analogous to the term *Simpson's ubiquity* for the compact case (cf. [Sim91]).

Theorem 2.8. — *Let X be a smooth quasi-projective variety. Consider $\mathfrak{C}$, a Zariski closed subset of $M_{\mathrm{B}}(X,G)(\mathbb{C})$, where G denotes a complex reductive group. If $\mathfrak{C}$ is invariant under the action of $\mathbb{R}^*$ defined above, then each geometrically connected component of $\mathfrak{C}(\mathbb{C})$ contains a $\mathbb{C}$-point $[\varrho]$ such that $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is a reductive representation that underlies a $\mathbb{C}$-variation of Hodge structure.* □

2.5. Pullback of reductive representations commutes with $\mathbb{C}^*$-action. — In this section, we prove that the $\mathbb{C}^*$-action on character varieties commutes with the pullback.

Proposition 2.9. — *Let $f : Y \to X$ be a morphism of smooth quasi-projective varieties. If $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is a reductive representation, then for any $t \in \mathbb{C}^*$, we have*

$$f^*(t.[\varrho]) = t.[f^*\varrho]. \tag{2.9}$$

Proof. — Let $\overline{X}$ and $\overline{Y}$ be smooth projective compactifications of X and Y such that $D := \overline{X}\backslash X$ and $D' := \overline{Y}\backslash Y$ are simple normal crossing divisors. We may assume that f extends to a morphism $f : \overline{Y} \to \overline{X}$.

By Theorem 1.6, there is a tame pure imaginary harmonic bundle (E,θ,h) on X such that ϱ is the monodromy representation of the flat connection $\nabla_h+\theta+\theta_h^\dagger$. Then $f^*\varrho$ is the monodromy representation of $f^*(\nabla_h+\theta+\theta_h^\dagger)$, which is the flat connection corresponding to the harmonic bundle $(f^*E, f^*\theta, f^*h)$.

Let $(\boldsymbol{E}_*,\theta)$ be the induced regular filtered Higgs bundle on $(\overline{X},D)$ by (E,θ,h) defined in § 2.1. According to §§ 2.2 and 2.3 we can define the pullback $(f^*\boldsymbol{E}_*, f^*\theta)$, which also forms a regular filtered Higgs bundle on $(\overline{Y},D')$ with trivial characteristic numbers.

Fix some ample line bundle L on $\overline{X}$. It is worth noting that for any $t \in \mathbb{C}^*$, $(\boldsymbol{E}_*, t\theta)$ is μ_L-polystable with trivial characteristic numbers. By [Moc06, Theorem 9.4], there is a pluriharmonic metric h_t for $(E,t\theta)$ adapted to the parabolic structures of $(\boldsymbol{E}_*,t\theta)$. Recall that in § 2.4, ϱ_t is defined to be the monodromy representation of the flat connection $\nabla_{h_t}+t\theta+\bar{t}\theta_{h_t}^\dagger$. It follows that $f^*\varrho_t$ is the monodromy representation of the flat connection $f^*(\nabla_{h_t}+t\theta+\bar{t}\theta_{h_t}^\dagger)$.

By virtue of Proposition 2.5, the regular filtered Higgs bundle $(f^*\boldsymbol{E}_*, tf^*\theta)$ is the prolongation of the tame harmonic bundle $(f^*E, tf^*\theta, f^*h_t)$ using norm growth defined in § 2.1. By the definition of $\mathbb{C}^*$-action, $(f^*\varrho)_t$ is the monodromy representation of the flat connection $\nabla_{f^*h_t}+tf^*\theta+\bar{t}(f^*\theta)_{f^*h_t}^\dagger$, which is equal to $f^*(\nabla_{h_t}+t\theta+\bar{t}\theta_{h_t}^\dagger)$. It follows that $(f^*\varrho)_t = f^*\varrho_t$. This concludes (2.9). □

As a direct consequence of Proposition 2.9, we have the following result.

Corollary 2.10. — *Let $f : Y \to X$ be a morphism of smooth quasi-projective varieties. Let $M \subset M_{\mathrm{B}}(X,N)(\mathbb{C})$ be a subset which is invariant by $\mathbb{C}^*$-action (or $\mathbb{R}^*$-action). Then for the morphism $f^* : M_{\mathrm{B}}(X,N) \to M_{\mathrm{B}}(Y,N)$ between character varieties, f^*M is also invariant by $\mathbb{C}^*$-action (or $\mathbb{R}^*$-action).* □

3. Construction of the Shafarevich morphism

The aim of this section is to establish the proofs of Theorem A. Additionally, the techniques developed in this section will play a crucial role in § 4 dedicated to the proof of the reductive Shafarevich conjecture.

3.1. Factorizing through non-rigidity. — In this subsection, X is assumed to be a smooth quasi-projective variety. Let $\mathfrak{C} \subset M_{\mathrm{B}}(X,N)(\mathbb{C})$ be a $\bar{\mathbb{Q}}$-constructible subset. Since $M_{\mathrm{B}}(X,N)$ is an finite type affine scheme defined over $\mathbb{Q}$, $\mathfrak{C}$ is defined over some number field k.

Let us utilize Lemma 1.29 and Theorem 1.27 to construct a reduction map $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ associated with $\mathfrak{C}$, which allows us to factorize non-rigid representations into those underlying $\mathbb{C}$-VHS with discrete monodromy.

Definition 3.1. — The *reduction map* $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ is obtained through the simultaneous Stein factorization of the reductions $\{s_\tau : X \to S_\tau\}_{[\tau]\in\mathfrak{C}(K)}$, employing Lemma 1.29. Here $\tau : \pi_1(X) \to \mathrm{GL}_N(K)$ ranges over all reductive representations with K a non-archimedean local field containing k such that $[\tau] \in \mathfrak{C}(K)$ and $s_\tau : X \to S_\tau$ is the reduction map constructed in Theorem 1.27.

Note that $s_{\mathfrak{C}}$ is a dominant morphism with connected general fibers and we have the following diagram.

$$\begin{tikzcd} X \arrow[r, "s_{\mathfrak{C}}"] \arrow[dr, "s_\tau"'] & S_{\mathfrak{C}} \arrow[d, "e_\tau"] \\ & S_\tau \end{tikzcd}$$

The reduction map $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ employs the following crucial property, thanks to Theorem 1.27.

Lemma 3.2. — *Let $F \subset X$ be a connected Zariski closed subset such that $s_{\mathfrak{C}}(F)$ is a single point in $S_{\mathfrak{C}}$. Then for any non-archimedean local field L and any reductive representation $\tau : \pi_1(X) \to \mathrm{GL}_N(L)$, the image $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is a bounded subgroup of $\mathrm{GL}_N(L)$.*

Proof. — By our construction $s_\tau = e_\tau \circ s_{\mathfrak{C}}$, so $s_\tau(F)$ is a single point. Hence by Theorem 1.27, $\tau(\mathrm{Im}[\pi_1(F) \to \pi_1(X)])$ is bounded. □

Recall the following definition in [KP23, Definition 2.2.1].

***Definition 3.3* (Bounded set).** — Let K be a non-archimedean local field. Let X be an affine K-scheme. A subset $B \subset X(K)$ is *bounded* if for every $f \in K[X]$, the set $\{v(f(b)) \mid b \in B\}$ is bounded below, where $v : K \to \mathbb{R}$ is the valuation of K.

We have the following lemma in [KP23, Fact 2.2.3].

Lemma 3.4. — *If $B \subset X(K)$ is closed, then B is bounded if and only if B is compact with respect to the analytic topology of $X(K)$. If $f : X \to Y$ is a morphism of affine K-schemes of finite type, then f carries bounded subsets of $X(K)$ to bounded subsets in $Y(K)$.* □

We will establish a lemma that plays a crucial role in the proof of Proposition 3.9 and is also noteworthy in its own regard.

Lemma 3.5. — *Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ be a (un)bounded representation. Then its semisimplification $\varrho^{ss} : \pi_1(X) \to \mathrm{GL}_N(\bar{K})$ is also (un)bounded.*

Proof. — Note that there exists some $g \in \mathrm{GL}_N(\bar{K})$ such that

$$g\varrho g^{-1} = \begin{bmatrix} \varrho_1 & a_{12} & \cdots & a_{1n} \\ 0 & \varrho_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varrho_n \end{bmatrix} \tag{3.1}$$

where $\varrho_i : \pi_1(X) \to \mathrm{GL}_{N_i}(\bar{K})$ is an irreducible representation such that $\sum_{i=1}^N N_i = N$. Note that $g\varrho g^{-1}$ is unbounded if and only if ϱ is unbounded. Hence we may assume at the beginning that ϱ has the form of (3.1). The semisimplification of ϱ is defined by

$$\varrho^{ss} = \begin{bmatrix} \varrho_1 & 0 & \cdots & 0 \\ 0 & \varrho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varrho_n \end{bmatrix}$$

It is obvious that if ϱ is bounded, then ϱ^{ss} is bounded.

Assume now ϱ^{ss} is bounded. Then each ϱ_i is bounded. Let L be a finite extension of K such that ϱ is defined over L. Then $\varrho_i(\pi_1(X))$ is contained in some maximal compact subgroup of $\mathrm{GL}_{N_i}(L)$. Since all maximal compact subgroups of $\mathrm{GL}_{N_i}(L)$ are conjugate to $\mathrm{GL}_{N_i}(\mathcal{O}_L)$, then there exists $g_i \in \mathrm{GL}_{N_i}(L)$ such that $g_i\varrho_i g_i^{-1} : \pi_1(X) \to \mathrm{GL}_{N_i}(\mathcal{O}_L)$. Define

$$\tau := \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_n \end{bmatrix} \begin{bmatrix} \varrho_1 & a_{12} & \cdots & a_{1n} \\ 0 & \varrho_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varrho_n \end{bmatrix} \begin{bmatrix} g_1^{-1} & 0 & \cdots & 0 \\ 0 & g_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_n^{-1} \end{bmatrix} \tag{3.2}$$

which is conjugate to ϱ, and is thus unbounded. Then τ can be written as

$$\tau = \begin{bmatrix} g_1\varrho_1g_1^{-1} & h_{12} & \cdots & h_{1n} \\ 0 & g_2\varrho_2g_2^{-1} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_n\varrho_ng_n^{-1} \end{bmatrix}$$

such that $g_i\varrho_ig_i^{-1} : \pi_1(X) \to \mathrm{GL}_N(\mathcal{O}_L)$ is irreducible. Write

$$\tau_1 := \begin{bmatrix} g_1\varrho_1g_1^{-1} & 0 & \cdots & 0 \\ 0 & g_2\varrho_2g_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_n\varrho_ng_n^{-1} \end{bmatrix}$$

and

$$\tau_2 := \begin{bmatrix} 0 & h_{12} & \cdots & h_{1n} \\ 0 & 0 & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Note that τ_2 is not a group homomorphism but only a map from $\pi_1(X)$ to $\mathrm{GL}_N(L)$.

For any matrix B with values in L, we shall write $v(B)$ the matrix whose entries are the valuation of the corresponding entries in B by $v : L \to \mathbb{R}$. Let us define $M(B)$ the lower bound of the entries of $v(B)$. Then for another matrix A with values in L, one has $M(A+B) \geq \min\{M(A), M(B)\}$.

Let $x_1, \ldots, x_m$ be a generator of $\pi_1(X)$. Let C be the lower bound of the entries of $v(h_{ij}(x_k))$. We assume that $C < 0$, or else it is easy to see that τ is bounded. Note that $\min_{i=1,\ldots,m} M(g_i\varrho_ig_i^{-1}(x_i)) \geq 0$. It follows that $M(\tau_1(x_i)) \geq 0$ for each x_i. Then for any $x = x_{i_1} \cdots x_{i_\ell}$,

$$\begin{aligned} M(\tau(x)) &= M\Big(\sum_{j_1,\ldots,j_\ell=1,2} \tau_{j_1}(x_{i_1}) \cdots \tau_{j_\ell}(x_{i_\ell})\Big) \\ &\geq \min_{j_1,\ldots,j_\ell=1,2} \{M(\tau_{j_1}(x_{i_1}) \cdots \tau_{j_\ell}(x_{i_\ell}))\}. \end{aligned}$$

Note that $\tau_{j_1}(x_{i_1}) \cdots \tau_{j_\ell}(x_{i_\ell}) = 0$ if $\#\{k \mid j_k = 2\} \geq n$ since $\tau_2(x_i)$ is nipotent. Hence

$$M(\tau(x)) \geq \min_{j_1,\ldots,j_\ell=1,2;\#\{k|j_k=2\}<n} \{M(\tau_{j_1}(x_{i_1}) \cdots \tau_{j_\ell}(x_{i_\ell}))\}.$$

Since $M(\tau_1(x_i)) \geq 0$ for each x_i, it follows that $M(\tau_{j_1}(x_{i_1}) \cdots \tau_{j_\ell}(x_{i_\ell}) \geq (n-1)C$ if $\#\{k \mid j_k = 2\} < n$. Therefore, $M(\tau(x)) \geq (n-1)C$ for any $x \in \pi_1(X)$. τ is thus bounded. Since ϱ is conjugate to τ, ϱ is also bounded. We finish the proof of the lemma. □

We recall the following facts of character varieties.

Lemma 3.6. — *Let K be an* algebraically closed *field of characteristic zero. Then the K-points $M_{\mathrm{B}}(X, N)$ are in one-to-one correspondence with the conjugate classes of reductive representations $\pi_1(X) \to \mathrm{GL}_N(K)$. More precisely, if $\{\varrho_i : \pi_1(X) \to \mathrm{GL}_N(K)\}_{i=1,2}$ are two linear representations such that $[\varrho_1] = [\varrho_2] \in M_{\mathrm{B}}(X, N)(K)$, then the semisimplification of ϱ_1 and ϱ_2 are conjugate.* □

The following result is thus a consequence of Lemma 3.5.

Lemma 3.7. — *Let K be a non-archimedean local field. Let $x \in M_{\mathrm{B}}(X,N)(K)$. If $\{\varrho_i : \pi_1(X) \to \mathrm{GL}_N(\bar{K})\}_{i=1,2}$ are two linear representations such that $[\varrho_1] = [\varrho_2] = x \in M_{\mathrm{B}}(X,N)(\bar{K})$, then ϱ_1 is bounded if and only if ϱ_2 is bounded. In other words, for the GIT quotient $\pi : R(X,N) \to M_B(X,N)$ where $R(X,N)$ is the representation variety of $\pi_1(X)$ into GL_N, for any $x \in M_B(X,N)(\bar{K})$, the representations in $\pi^{-1}(x) \subset R(X,N)(\bar{K})$ are either all bounded or all unbounded.*

Proof. — By the assumption and Lemma 3.6, we know that the semisimplificaitons $\varrho_1^{ss} : \pi_1(X) \to \mathrm{GL}_N(\bar{K})$ of $\varrho_2^{ss} : \pi_1(X) \to \mathrm{GL}_N(\bar{K})$ are conjugate by an element $g \in \mathrm{GL}_N(\bar{K})$. Therefore, there exists a finite extension L of K such that ϱ_i^{ss} and ϱ_i are all defined in L and $g \in \mathrm{GL}_N(L)$. Hence ϱ_1^{ss} is bounded if and only if ϱ_2^{ss} is bounded. By Lemma 3.5, we know that ϱ_i^{ss} is bounded if and only if ϱ_i is bounded. Therefore, the lemma follows. □

We thus can make the following definition.

***Definition 3.8* (Class of bounded representations).** — Let K be a non-archimedean local field of characteristic zero. A point $x \in M_B(X,N)(\bar{K})$ is called *a class of bounded representations* if there exist $\varrho : \pi_1(X) \to \mathrm{GL}_N(\bar{K})$ (thus any ϱ by Lemma 3.7) such that $[\varrho] = x$ and ϱ is bounded.

Proposition 3.9. — *Let X be a smooth quasi-projective variety and let $\mathfrak{C}$ be a $\bar{\mathbb{Q}}$-constructible subset of $M_{\mathrm{B}}(X,N)$. Let $\iota : F \to X$ be a morphism from a quasi-projective* normal *variety F such that $s_{\mathfrak{C}} \circ \iota(F)$ is a point. Let $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,2}$ be reductive representations such that $[\tau_1]$ and $[\tau_2]$ are in the same geometric connected component of $\mathfrak{C}(\mathbb{C})$. Then $\tau_1 \circ \iota$ is conjugate to $\tau_2 \circ \iota$. In other words, $j(\mathfrak{C})$ is zero-dimensional, where $j : M_{\mathrm{B}}(X,N) \to M_{\mathrm{B}}(F,N)$ is the natural morphism of character varieties induced by $\iota : F \to X$.*

Proof. — Let M_X (resp. M) be the moduli space of representations of $\pi_1(X)$ (resp. $\pi_1(F^{\mathrm{norm}})$) in GL_N. Note that M_X and M are both affine schemes of finite type defined over $\mathbb{Q}$. Let R_X (resp. R) be the affine scheme of finite type defined over $\mathbb{Q}$ such that $R_X(L) = \mathrm{Hom}(\pi_1(X), \mathrm{GL}_N(L))$ (resp. $R(L) = \mathrm{Hom}(\pi_1(F), \mathrm{GL}_N(L))$) for any field $L/\mathbb{Q}$. Then we have

$$\begin{tikzcd} R_X \arrow[r, "\pi"] \arrow[d, "\iota^*"] & M_X \arrow[d, "j"] \\ R \arrow[r, "p"] & M \end{tikzcd} \tag{3.3}$$

where $\pi : R_X \to M_X$ and $p : R \to M$ are the GIT quotient that are both surjective. For any field extension $K/\mathbb{Q}$ and any $\varrho \in R_X(K)$, we write $[\varrho] := \pi(\varrho) \in M_X(K)$. Let $\mathfrak{R} := \pi^{-1}(\mathfrak{C})$ that is a constructible subset defined over some number field k. Then $\tau_i \in \mathfrak{R}(\mathbb{C})$.

Claim 3.10. — *Let $\mathfrak{R}'$ be any geometric irreducible component of $\mathfrak{R}$. Then $j \circ \pi(\mathfrak{R}')$ is zero dimensional.*

Proof. — Assume, for the sake of contradiction, that $j \circ \pi(\mathfrak{R}')$ is positive-dimensional. If we replace k by a finite extension, we may assume that $\mathfrak{R}'$ is defined over k. Since M is an affine $\mathbb{Q}$-scheme of finite type, it follows that there exist a k-morphism $\psi : M \to \mathbb{A}^1$ such that the image $\psi \circ j \circ \pi(\mathfrak{R}')$ is Zariski dense in $\mathbb{A}^1$. After replacing k by a finite extension, we can find a locally closed irreducible curve $C \subset \mathfrak{R}'$ such that the restriction $\psi \circ j \circ \pi|_C : C \to \mathbb{A}^1$ is a generically finite k-morphism. We take a Zariski open subset $U \subset \mathbb{A}^1$ such that $\psi \circ j \circ \pi$ is finite over U. Let $\mathfrak{p}$ be a prime ideal of the ring of integer $\mathcal{O}_k$ and let K be its non-archimedean completion. In the following, we shall work over K.

Let $x \in U(K)$ be a point, and let $y \in C(\bar{K})$ be a point over x. Then y is defined over some extension of K whose extension degree is bounded by the degree of $\psi \circ j \circ \pi|_C : C \to \mathbb{A}^1$. Note that there are only finitely many such field extensions. Hence there exists a finite extension L/K such that the points over $U(K)$ are all contained in $C(L)$. Since $U(K) \subset \mathbb{A}^1(L)$ is unbounded, the image $\psi \circ j \circ \pi(C(L)) \subset \mathbb{A}^1(L)$ is unbounded.

Write $p : R \to M$ be the GIT quotient. Let R_0 be the set of bounded representations in $R(L)$. Recall that by [Yam10], $M_0 := p(R_0)$ is compact in $M(L)$ with respect to analytic topology, hence M_0 is bounded by Lemma 3.4. By Lemma 3.4 once again, $\psi(M_0)$ is a bounded subset in $\mathbb{A}^1(L)$. Recall that $\psi \circ j \circ \pi(C(L)) \subset \mathbb{A}^1(L)$ is unbounded. Therefore, there exists $\varrho \in C(L)$ such that $\psi \circ j([\varrho]) \notin \psi(M_0)$. Note that $[\varrho \circ \iota] = j([\varrho])$ by (3.3). Hence $[\varrho \circ \iota] \notin M_0$ which implies that $\varrho \circ \iota \notin R_0$. By definition of R_0, $\varrho \circ \iota$ is unbounded.

Let $\varrho^{ss} : \pi_1(X) \to \mathrm{GL}_N(\bar{L})$ be the semisimplification of ϱ. Then $[\varrho] = [\varrho^{ss}] \in \mathfrak{C}(\bar{L})$ by Lemma 3.6. Therefore, $[\varrho \circ \iota] = [\varrho^{ss} \circ \iota] \in M(\bar{L})$ by (3.3). By Lemma 3.7, $\varrho^{ss} \circ \iota : \pi_1(F) \to \mathrm{GL}_N(\bar{L})$ is also unbounded. Note that $\varrho^{ss} \circ \iota$ is reductive by Theorem 1.7. Since $\pi_1(F)$ is finitely generated, there exist a finite extension L' of L such that ϱ^{ss} is defined over L'. However, by Lemma 3.2, $\varrho^{ss} \circ \iota$ is always bounded. We obtain a contradiction and thus $j \circ \pi(\mathfrak{R}')$ is zero dimensional.

We can also apply [Kem78, Corollary 4.3] instead of Lemma 3.7. As $\varrho \in C(L)$, its image $[\varrho] \in M_X(L)$. Consider the fiber $\pi^{-1}([\varrho])$ which is a L-variety. Its closed orbit is defined over L by Galois descent. As $\pi^{-1}([\varrho])$ contains the L-point ϱ, the closed orbit in $\pi^{-1}([\varrho])$ has an L-point $\varrho' : \pi_1(X) \to \mathrm{GL}_N(L)$ as well by [Kem78, Corollary 4.3]. By Lemma 3.6 ϱ' is reductive and $[\varrho'] = [\varrho]$. Hence $[\varrho' \circ \iota] = [\varrho \circ \iota] \notin M_0$. Therefore, $\varrho' \circ \iota : \pi_1(F) \to \mathrm{GL}_N(L)$ is unbounded by our definition of M_0. However, by the definition of $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ in Definition 3.1, $\varrho' \circ \iota$ is always bounded. We obtain a contradiction and thus $j \circ \pi(\mathfrak{R}')$ is zero dimensional. □

Let $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,2}$ be reductive representations such that $[\tau_1]$ and $[\tau_2]$ are contained in the same connected component $\mathfrak{C}'$ of $\mathfrak{C}(\mathbb{C})$. We aim to prove that $j(\mathfrak{C}')$ is a point in $M(\mathbb{C})$.

Consider an irreducible component $\mathfrak{C}''$ of $\mathfrak{C}'$. We can choose an irreducible component Z of $\pi^{-1}(\mathfrak{C}'')$ such that $\pi(Z)$ is dense in $\mathfrak{C}''$. It follows that Z is an irreducible component of $\mathfrak{R}(\mathbb{C})$. By Claim 3.10, we know that $j \circ \pi(Z)$ is a point in $M(\mathbb{C})$. Thus, $j(\mathfrak{C}'')$ is also a point in $M(\mathbb{C})$.

Consequently, $j(\mathfrak{C}')$ is a point in $M(\mathbb{C})$. As a result, we have $[\tau_1 \circ \iota] = j([\tau_1]) = j([\tau_2]) = [\tau_2 \circ \iota]$. By Theorem 1.7, $\tau_1 \circ \iota$ and $\tau_2 \circ \iota$ are reductive, and according to Lemma 3.6, they are conjugate to each other. We have established the proposition.

□

We will need the following lemma on the intersection of kernels of representations.

Lemma 3.11. — *Let X be a quasi-projective normal variety and let $\mathfrak{C}$ be a constructible subset of $M_{\mathrm{B}}(X, N)(\mathbb{C})$. Then we have*

$$\cap_{[\varrho] \in \mathfrak{C}} \ker \varrho = \cap_{[\varrho] \in \overline{\mathfrak{C}}} \ker \varrho, \tag{3.4}$$

where ϱ's are reductive representations of $\pi_1(X)$ into $\mathrm{GL}_N(\mathbb{C})$.

Proof. — Let M_X be the moduli space of representation of $\pi_1(X)$ in GL_N. Let R_X be the affine scheme of finite type such that $R(L) = \mathrm{Hom}(\pi_1(X), N)(L)$ for any field $\mathbb{Q} \subset L$. We write $M := M_X(\mathbb{C})$ and $R := R_X(\mathbb{C})$. Then the GIT quotient $\pi : R \to M$ is a surjective morphism. It follows that $\pi^{-1}(\mathfrak{C})$ is a $\mathrm{GL}_N(\mathbb{C})$-invariant subset where $\mathrm{GL}_N(\mathbb{C})$ acts on R by the conjugation. Define $H := \cap_{[\varrho] \in \mathfrak{C}} \ker \varrho$, where ϱ's are reductive representations of $\pi_1(X)$ into $\mathrm{GL}_N(\mathbb{C})$. Pick any $\gamma \in H$. Then the set $Z_\gamma := \{\varrho \in R \mid \varrho(\gamma) = 1\}$ is a Zariski closed subset of R. Moreover, Z_γ is $\mathrm{GL}_N(\mathbb{C})$-invariant. Define $Z := \cap_{\gamma \in H} Z_\gamma$. Then Z is also $\mathrm{GL}_N(\mathbb{C})$-invariant. Therefore, $\pi(Z)$ is also a Zariski closed subset of M. Note that $\mathfrak{C} \subset \pi(Z)$. Therefore, $\overline{\mathfrak{C}} \subset \pi(Z)$. Note that for any reductive $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in \pi(Z)$, we have $\varrho(\gamma) = 1$ for any $\gamma \in H$. It follows that (3.4) holds. □

Lastly, let us prove the main result of this subsection. This result will serve as a crucial cornerstone in the proofs of Theorems A to C.

Proposition 3.12. — *Let X be a smooth quasi-projective variety. Let $\mathfrak{C}$ be a constructible subset of $M_{\mathrm{B}}(X, N)(\mathbb{C})$, defined over $\mathbb{Q}$, such that $\mathfrak{C}$ is invariant under $\mathbb{R}^*$-action. When*

X is non-compact, we further assume that $\mathfrak{C}$ is closed. Then there exist reductive representations $\{\sigma_i^{VHS}:\pi_1(X)\to \mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,m}$ such that each σ_i^{VHS} underlies a $\mathbb{C}$-VHS, and for a morphism $\iota:Z\to X$ from any quasi-projective normal *variety Z with $s_{\mathfrak{C}}\circ\iota(Z)$ being a point, the following properties hold:*

(i) *For $\sigma:=\oplus_{i=1}^m\sigma_i^{VHS}$, $\iota^*\sigma(\pi_1(Z))$ is discrete in $\prod_{i=1}^m\mathrm{GL}_N(\mathbb{C})$.*
(ii) *For each reductive representation $\tau:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ with $[\tau]\in\mathfrak{C}(\mathbb{C})$, $\iota^*\tau$ is conjugate to some $\iota^*\sigma_i^{VHS}$.*
(iii) *For each σ_i^{VHS}, there exists a reductive representation $\tau:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ with $[\tau]\in\mathfrak{C}(\mathbb{C})$ such that $\iota^*\tau$ is conjugate to $\iota^*\sigma_i^{VHS}$.*
(iv) *For every $i=1,\ldots,m$, we have*

$$\cap_{[\varrho]\in\mathfrak{C}}\ker\varrho\subset\ker\sigma_i^{VHS}\tag{3.5}$$

where $\varrho:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ varies among all reductive representations such that $[\varrho]\in\mathfrak{C}(\mathbb{C})$.

Proof. — Let $\mathfrak{C}_1,\ldots,\mathfrak{C}_\ell$ be all geometric connected components of $\mathfrak{C}$ which are defined over $\bar{\mathbb{Q}}$. We can pick reductive representations $\{\varrho_i:\pi_1(X)\to\mathrm{GL}_N(\bar{\mathbb{Q}})\}_{i=1,\ldots,\ell}$ such that $[\varrho_i]\in\mathfrak{C}_i(\bar{\mathbb{Q}})$ for every i. Since $\pi_1(X)$ is finitely generated, there exists a number field k which is a Galois extension of $\mathbb{Q}$ such that $\varrho_i:\pi_1(X)\to\mathrm{GL}_N(k)$ for every ϱ_i.

Let $\mathrm{Ar}(k)$ be all archimedean places of k with w_1 the identity map. Then for any $w\in\mathrm{Ar}(k)$ there exists $a\in\mathrm{Gal}(k/\mathbb{Q})$ such that $w=w_1\circ a$. Note that $\mathfrak{C}$ is defined over $\mathbb{Q}$. Then $\mathfrak{C}$ is invariant under the conjugation a. Therefore, for any $w:k\to\mathbb{C}$ in $\mathrm{Ar}(k)$, letting $\varrho_{i,w}:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ be the composition $w\circ\varrho_i$, we have $[\varrho_{i,w}]\in\mathfrak{C}(\mathbb{C})$.

For any $t\in\mathbb{R}^*$, we consider the $\mathbb{R}^*$-action $\varrho_{i,w,t}:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ of $\varrho_{i,w}$ defined in § 2.4. Then $\varrho_{i,w,t}$ is also reductive by the arguments in § 2.4. Since we assume that $\mathfrak{C}(\mathbb{C})$ is invariant under $\mathbb{R}^*$-action, it follows that $[\varrho_{i,w,t}]\in\mathfrak{C}(\mathbb{C})$. By Lemma 2.7, $[\varrho_{i,w,t}]$ is a continuous deformation of $[\varrho_{i,w}]$. Hence they are in the same geometric connected component of $\mathfrak{C}(\mathbb{C})$, and by Proposition 3.9 we conclude that $[\iota^*\varrho_{i,w,t}]=[\iota^*\varrho_{i,w}]$ for any $t\in\mathbb{R}^*$.

We first assume that X is compact. According to [Sim92], $\lim_{t\to 0}[\varrho_{i,w,t}]$ exists, and there exists a reductive $\varrho_{i,w}^{VHS}:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ such that $[\varrho_{i,w}^{VHS}]=\lim_{t\to 0}[\varrho_{i,w,t}]$. Moreover, $\varrho_{i,w}^{VHS}$ underlies a $\mathbb{C}$-VHS. Therefore, $[\iota^*\varrho_{i,w}]=\lim_{t\to 0}[\iota^*\varrho_{i,w,t}]=[\iota^*\varrho_{i,w}^{VHS}]$. Since $[\varrho_{i,w,t}]\in\mathfrak{C}(\mathbb{C})$ for any $t\in\mathbb{R}^*$, it follows that $[\varrho_{i,w}^{VHS}]\in\overline{\mathfrak{C}}(\mathbb{C})$. By eq. (3.4), we conclude

$$\cap_{[\varrho]\in\mathfrak{C}}\ker\varrho\subset\ker\varrho_{i,w}^{VHS}.\tag{3.6}$$

Assume now X is non-compact. As we assume that $\mathfrak{C}$ is closed and invariant under $\mathbb{R}^*$-action, by Theorem 2.8, we can choose a reductive representation $\varrho_{i,w}^{VHS}:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ such that

— it underlies a $\mathbb{C}$-VHS;
— $[\varrho_{i,w}^{VHS}]$ and $[\varrho_{i,w}]$ are in the same geometric connected component of $\mathfrak{C}(\mathbb{C})$.

Note that (3.6) is satisfied automatically. By Proposition 3.9, we have $[\iota^*\varrho_{i,w}]=[\iota^*\varrho_{i,w}^{VHS}]$.

In summary, we construct reductive representations $\{\varrho_{i,w}^{VHS}:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,k;w\in\mathrm{Ar}(k)}$ in both compact and non-compact cases. Each of these representations underlies a $\mathbb{C}$-VHS and satisfies $[\iota^*\varrho_{i,w}]=[\iota^*\varrho_{i,w}^{VHS}]$ and (3.6).

Let v be any non-archimedean place of k and k_v be the non-archimedean completion of k with respect to v. Write $\varrho_{i,v}:\pi_1(X)\to\mathrm{GL}_N(k_v)$ the induced representation from ϱ_i. By the construction of $s_{\mathfrak{C}}$, it follows that $\iota^*\varrho_{i,v}(\pi_1(Z))$ is bounded. Therefore, we have a factorization

$$\iota^*\varrho_i:\pi_1(Z)\to\mathrm{GL}_N(\mathcal{O}_k).$$

Note that $\mathrm{GL}_N(\mathcal{O}_k) \to \prod_{w\in \mathrm{Ar}(k)} \mathrm{GL}_N(\mathbb{C})$ is a discrete subgroup by [Zim84, Proposition 6.1.3]. It follows that for the product representation

$$\prod_{w\in \mathrm{Ar}(k)} \iota^*\varrho_{i,w} : \pi_1(Z) \to \prod_{w\in \mathrm{Ar}(k)} \mathrm{GL}_N(\mathbb{C}),$$

its image is discrete.

Since Z is normal, by Theorem 1.7, both $\iota^*\varrho_{i,w}$ and $\iota^*\varrho_{i,w}^{VHS}$ are reductive. Recall that $[\iota^*\varrho_{i,w}] = [\iota^*\varrho_{i,w}^{VHS}]$, it follows that $\iota^*\varrho_{i,w}$ is conjugate to $\iota^*\varrho_{i,w}^{VHS}$ by Lemma 3.6. Consequently, $\prod_{w\in \mathrm{Ar}(k)} \iota^*\varrho_{i,w}^{VHS} : \pi_1(Z) \to \mathrm{GL}_N(\mathbb{C})$ has discrete image. Consider the product representation of $\varrho_{i,w}^{VHS}$

$$\sigma := \prod_{i=1}^{\ell}\prod_{w\in \mathrm{Ar}(k)} \varrho_{i,w}^{VHS} : \pi_1(X) \to \prod_{i=1}^{\ell}\prod_{w\in \mathrm{Ar}(k)} \mathrm{GL}_N(\mathbb{C}).$$

Then σ underlies a $\mathbb{C}$-VHS and $\iota^*\sigma : \pi_1(Z) \to \prod_{i=1}^{\ell}\prod_{w\in \mathrm{Ar}(k)} \mathrm{GL}_N(\mathbb{C})$ has discrete image.

Let $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be any reductive representation such that $[\tau] \in \mathfrak{C}(\mathbb{C})$. Then $[\tau] \in \mathfrak{C}_i(\mathbb{C})$ for some i. By Proposition 3.9, it follows that $[\iota^*\tau] = [\iota^*\varrho_{i,w_1}] = [\iota^*\varrho_{i,w_1}^{VHS}]$. By Theorem 1.7 and Lemma 3.6 once again, $\iota^*\tau$ is conjugate to $\iota^*\varrho_{i,w_1}^{VHS}$. The proposition is proved if we let $\{\sigma_i^{VHS} : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,m}$ be $\{\varrho_{i,w}^{VHS} : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,\ell; w\in \mathrm{Ar}(k)}$. □

Remark 3.13. — In the proof of Proposition 3.12, we take the Galois conjugate of $\mathfrak{C} \subset M_{\mathrm{B}}(X,N)$ under $a \in \mathrm{Gal}(k/\mathbb{Q})$. If $\mathfrak{C}$ is not defined over $\mathbb{Q}$, it is not known that $a(\mathfrak{C}) \subset M_{\mathrm{B}}(X,N)$ is $\mathbb{R}^*$-invariant. This is why we include the assumption that $\mathfrak{C}$ is defined over $\mathbb{Q}$ in our proof, whereas Eyssidieux disregarded such a condition in [Eys04]. It seems that this condition should also be necessary in [Eys04].

3.2. Infinite monodromy at infinity. — When considering a non-compact quasi-projective variety X, it is important to note that the Shafarevich conjecture fails in simple examples. For instance, take $X := A\backslash\{0\}$, where A is an abelian surface. Its universal covering $\widetilde{X}$ is $\mathbb{C}^2 - \Gamma$, where Γ is a lattice in $\mathbb{C}^2$. Then $\widetilde{X}$ is not holomorphically convex. Therefore, additional conditions on the fundamental groups at infinity are necessary to address this issue.

Definition 3.14 **(Infinity monodromy at infinity).** — Let X be a quasi-projective normal variety and let $\overline{X}$ be a projective compactification of X. We say a subset $M \subset M_{\mathrm{B}}(X,N)(\mathbb{C})$ *has infinite monodromy at infinity* if for any holomorphic map $\gamma : \mathbb{D} \to \overline{X}$ with $\gamma^{-1}(\overline{X} \setminus X) = \{0\}$, there exists a reductive $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in M$ and $\gamma^*\varrho : \pi_1(\mathbb{D}^*) \to \mathrm{GL}_N(\mathbb{C})$ has infinite image.

Note that Definition 3.14 does not depend on the projective compactification of X.

Lemma 3.15. — *Let $f : Y \to X$ be a* proper *morphism between quasi-projective normal varieties. If $M \subset M_{\mathrm{B}}(X,N)(\mathbb{C})$ has infinite monodromy at infinity, then $f^*M \subset M_{\mathrm{B}}(Y,N)(\mathbb{C})$ also has infinite monodromy at infinity.*

Proof. — We take projective compactification $\overline{X}$ and $\overline{Y}$ of X and Y respectively such that f extends to a morphism $\bar{f} : \overline{Y} \to \overline{X}$. Let $\gamma : \mathbb{D} \to \overline{Y}$ be any holomorphic map with $\gamma^{-1}(\overline{Y} \setminus Y) = \{0\}$. Then $\bar{f} \circ \gamma : \mathbb{D} \to \overline{X}$ satisfies $(\bar{f} \circ \gamma)^{-1}(\overline{X} \setminus X) = \{0\}$ as f is proper. Then by Definition 3.14 there exists a reductive $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in M$ and $\gamma^*(f^*\varrho) = (f \circ \gamma)^*\varrho : \pi_1(\mathbb{D}^*) \to \mathrm{GL}_N(\mathbb{C})$ has infinite image. The lemma follows. □

We have a precise local characterization of a representation with infinite monodromy at infinity.

Lemma 3.16. — *Consider a smooth quasi-projective variety X along with a smooth projective compactification $\overline{X}$, where $D := \overline{X}\backslash X$ is a simple normal crossing divisor. A set $M \subset M_{\mathrm{B}}(X,N)(\mathbb{C})$ has infinite monodromy at infinity is equivalent to the following:*

for any $x \in D$, there exists an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x with $U \cap D = (z_1 \cdots z_k = 0)$ such that for any k-tuple $(i_1, \ldots, i_k) \in \mathbb{Z}^k_{>0}$, there exists a reductive $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in M(\mathbb{C})$ and $\varrho(\gamma_1^{i_1} \cdots \gamma_k^{i_k}) \neq 0$, where γ_i is the anti-clockwise loop around the origin in the i-th factor of $U \setminus D \simeq (\mathbb{D}^*)^k \times \mathbb{D}^{n-k}$. For such condition we will say that ϱ has infinite monodromy *at x.*

Proof. — For any holomorphic map $f : \mathbb{D} \to \overline{X}$ with $f^{-1}(D) = \{0\}$, let $x := f(0)$ which lies on D. We take an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x in the lemma. Then $f(\mathbb{D}_{2\varepsilon}) \subset U$ for some small $\varepsilon > 0$. We can write $f(t) = (f_1(t), \ldots, f_n(t))$ such that $f_1(0) = \cdots = f_k(0) = 0$ and $f_i(0) \neq 0$ for $i = k+1, \ldots, n$. Denote by $m_i := \mathrm{ord}_0 f_i$ the vanishing order of $f_i(t)$ at 0. Consider the anti-clockwise loop γ defined by $\theta \mapsto \varepsilon e^{i\theta}$ which generates $\pi_1(\mathbb{D}^*_{2\varepsilon})$. Then $f \circ \gamma$ is homotopy equivalent to $\gamma_1^{m_1} \cdots \gamma_k^{m_k}$ in $\pi_1(U \backslash D)$. If M has infinite monodromy at infinity, by Definition 3.14 there exists a reductive $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in M(\mathbb{C})$ and $f^*\varrho(\gamma) \neq 0$. This is equivalent to that $\varrho(\gamma_1^{m_1} \cdots \gamma_k^{m_k}) \neq 0$. The lemma is proved. □

Definition 3.14 presents a stringent condition that is not be practically applicable in many situations. To address this issue, we establish the following result:

Proposition 3.17. — *Assume that $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ is a reductive representation with $\varrho(\pi_1(X))$ is torsion free. Then we can find a birational morphism $\mu : \overline{X}_0 \to \overline{X}$ by taking a sequence of blowing-ups with smooth centers such that*

(i) *μ is an isomorphism over X;*
(ii) *there exists a Zariski open set $X' \subset \overline{X}_0$ containing X such that ϱ extends to a representation ϱ_0 over $\pi_1(X')$;*
(iii) *$\varrho_0 : \pi_1(X') \to \mathrm{GL}_N(\mathbb{C})$ has infinite monodromy at infinity.*

Proof. — Write $D = \sum_{i=1}^m D_i$ into sum of irreducible components. We first look at smooth points of D. If there exists some irreducible component D_1 of D such that the local monodromy of ϱ around D_1 is finite (which is thus trivial as $\varrho(\pi_1(X))$ is assumed to be torsion-free), then ϱ extends across the irreducible component of D_1. It follows that ϱ extends to a representation $\pi_1(\overline{X} \backslash \cup_{i=2}^m D_i) \to \mathrm{GL}_N(\mathbb{C})$. We replace X by $\overline{X} \backslash \cup_{i=2}^m D_i$. To prove the proposition, we will use induction as follows.

We first define an index $i(x)$ of $x \in D$ by setting $i(x) := \#\{j \mid x \in D_j\}$. This depends on the compactification of X, and the index is computed with respect to the new boundary divisor if we blow-up the boundary D.

Induction. Assume that there exists an algorithm of the blowing-ups $\overline{X}_0 \to \overline{X}$ required in the proposition such that we can extend ϱ on some Zariski dense open set X' of $\overline{X}_0$ containing X, and achieve the following: for any point x in the new boundary $\overline{X}_0 \backslash X'$, ϱ has infinite monodromy at x if $i(x) \leq k-1$. Here the index of x is computed with respect to the new boundary divisor $\overline{X}_0 \backslash X'$.

We know that $k = 2$ can be achieved by the above argument without blowing-up $\overline{X}$.

By induction, we can assume for any $x \in D$ with $i(x) \leq k-1$, ϱ always has infinite monodromy at x in the sense of Lemma 3.16.

We will work on points in D of index k at which ϱ has infinite monodromy. We need to cover $D_1 \cup \ldots \cup D_m$ by a natural stratification. For any $J \subset \{1, \ldots, m\}$, define

$$D_J := \{x \in D_1 \cup \ldots \cup D_m \mid x \in D_j \Leftrightarrow j \in J\}.$$

Note that for any point $x \in D_J$, its index $i(x) = \#J$. It is worth noting that for any connected component Z of D_J, for each two points $x, y \in Z$, ϱ has infinite monodromy at x if and only if it has infinite monodromy at y. Therefore, we only have to deal with finitely many strata whose points have index is k.

Without loss of generality, we may assume that ϱ does not have infinite monodromy at the points of a connected component Z of the strata $D_{\{1,\ldots,k\}}$. Pick any point $x \in Z$. We choose an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x with $D_i \cap U = (z_i = 0)$

for $i=1,\ldots,k$ and $D_j\cap U=\varnothing$ for $j=k+1,\ldots,n$. Let γ_i be the anti-clockwise loop around the origin in the i-th factor of $U\backslash D\simeq (\mathbb{D}^*)^k\times \mathbb{D}^{n-k}$. By our assumption, there exists $(i_1,\ldots,i_k)\in \mathbb{Z}^k_{>0}$ such that $\varrho(\gamma_1^{i_1}\cdots\gamma_k^{i_k})=0$.

Claim 3.18. — *If there are some k-tuple $(j_1,\ldots,j_k)\in\mathbb{Z}^k_{>0}$ such that $\varrho(\gamma_1^{j_1}\cdots\gamma_k^{j_k})=0$, then $(j_1,\ldots,j_k)=\ell(i_1,\ldots,i_k)$ for some $\ell>0$.*

Proof. — Assume that the claim does not hold. After reordering $1,\ldots,k$, we can assume that

$$\frac{j_1}{i_1}=\cdots=\frac{j_{\ell-1}}{i_{\ell-1}}<\frac{j_\ell}{i_\ell}\leq\cdots\leq\frac{j_k}{i_k}$$

for some $\ell\in\{2,\ldots,k\}$. Then $i_1(j_1,\ldots,j_k)-j_1(i_1,\ldots,i_k)=(0,\cdots,0,i'_\ell,\cdots,i'_k)$ with $i'_\ell,\ldots,i'_k\in\mathbb{Z}_{>0}$. As $\varrho(\gamma_1^{i_1}\cdots\gamma_k^{i_k})=0$, it follows that $\varrho(\gamma_\ell^{i'_\ell}\cdots\gamma_k^{i'_k})=0$. Let us define a holomorphic map

$$g:\mathbb{D}\to U$$
$$t\mapsto(\frac{1}{2},\ldots,\frac{1}{2},t^{i'_\ell},\ldots,t^{i'_k}).$$

Then we have $1\leq i(g(0))\leq k-1$. The loop $\theta\mapsto g(\frac{1}{2}e^{i\theta})$ is homotopy to $\gamma_\ell^{i'_\ell}\cdots\gamma_k^{i'_k}$. Hence $g^*\varrho:\pi_1(\mathbb{D}^*)\to \mathrm{GL}_N(\mathbb{C})$ is trivial. As we assume that ϱ has infinite monodromy at $g(0)$, a contradiction is obtained. The claim follows. □

After reordering $1,\ldots,k$, we can assume that $i_1=\cdots=i_{\ell-1}<i_\ell\leq\ldots\leq i_k$. Then we have $2\leq\ell\leq k+1$. Here we make the convention that $i_1=\cdots=i_k$ if $\ell=k+1$. Since $\varrho(\pi_1(X))$ is torsion-free, we can replace the tuple $(i_1,\ldots,i_k)$ with $\frac{1}{\gcd(i_1,\ldots,i_k)}(i_1,\ldots,i_k)$. This allows us to assume that $\gcd(i_1,\ldots,i_k)=1$. Let $\overline{Z}$ be the closure of Z. Then it is a smooth, connected, closed subvariety of codimension k contained in $D_1\cap\ldots\cap D_k$. We proceed by performing the blow-up of $\overline{Z}$. Let D'_0 represent the exceptional divisor resulting from the blow-up, and we denote the strict transform of D_i as D'_i. It is important to note that $D'_1\cap\ldots\cap D'_k\cap D'_0=\varnothing$.

For any $J\subset\{1,\ldots,k\}$, we set

$$D'_J:=\left\{x\in D'_0\mid x\in D'_j\Leftrightarrow j\in J\right\},$$

and

$$D'_\varnothing:=D'_0\setminus(D'_1\cup\ldots\cup D'_k).$$

Note that for any $x\in D'_J$, its index $i(x)=1+\#J$. We can verify that

$$\mathcal{S}_k:=\{x\in D'_0\mid i(x)\leq k\}=\cup_{J\subset\{1,\ldots,k\}}D'_J.$$

Claim 3.19. — *For any point y in $\mathcal{S}_k$, ϱ has infinite monodromy at y if and only if $y\notin D_{\{\ell,\ldots,k\}}$. Here we make the convention that $\{\ell,\ldots,k\}=\varnothing$ if $\ell=k+1$.*

Proof. — Write $J=\{j_2,\ldots,j_p\}$. We make the convention that $J=\varnothing$ if $p=1$. Let $(V;w_1,\ldots,w_n)$ be an admissible coordinate centered at y with $V\cap D'_0=(w_1=0)$, $V\cap D'_{j_i}=(w_i=0)$ for $i=2,\ldots,p$ and $V\cap D'_q=\varnothing$ for other irreducible components D'_q of the boundary divisor. Let γ'_i be the anti-clockwise loop around the origin in the i-th factor of $(\mathbb{D}^*)^p\times\mathbb{D}^{n-p}$. We can see that $\gamma'_1\sim\gamma_1\cdots\gamma_k$, and $\gamma'_i\sim\gamma_{j_i}$ for $i=2,\ldots,p$. Here "$\sim$" stands for homotopy equivalent. Then for any p-tuple $(q_1,\ldots,q_p)\in\mathbb{Z}^p_{>0}$, writing $(\gamma'_1)^{q_1}\cdots(\gamma'_p)^{q_p}\sim\gamma_1^{n_1}\cdots\gamma_k^{n_k}$. An easy computation shows that $(n_1,\ldots,n_k)$ is never linear to $(i_1,\ldots,i_k)$. By Claim 3.18 we conclude that $\varrho((\gamma'_1)^{j_1}\cdots(\gamma'_k)^{j_k})\neq 0$ if $J\neq\{\ell,\ldots,k\}$. The claim is proved. □

By the above claim, there are two possibilites:
Case 1: $\#J<k-1$. In this case, for each $x\in\mathcal{S}_k$, one has $i(x)\leq k-1$. By our induction, we can perform a further sequence of blowing-ups with smooth centers in the boundary to obtain a birational morphism $\mu:\overline{X}'\to\overline{X}$ such that

(a) there exists a Zariski open set $X' \subset \overline{X}_0$ containing X such that ϱ extends to a representation ϱ_0 over $\pi_1(X')$;
(b) for any point $x \in \mu^{-1}(\overline{Z})$ with $i(x) \leq k$, ϱ_0 at infinite monodromy at x.

Case 2: $\#J = k-1$. In this case, $J = \{2, \ldots, k\}$. Pick any point $y \in D'_J$. Let $(V; w_1, \ldots, w_n)$ be an admissible coordinate centered at y with $V \cap D'_0 = (w_1 = 0)$ and $V \cap D'_j = (w_j = 0)$ for $j = 2, \ldots, k$. Let γ'_i be the anti-clockwise loop around the origin in the i-th factor of $(\mathbb{D}^*)^k \times \mathbb{D}^{n-k}$. We can see that $\gamma'_1 \sim \gamma_1 \cdots \gamma_k$, and $\gamma'_i \sim \gamma_i$ for $i = 2, \ldots, k$. Then for the k-tuple $(j_1, \ldots, j_k) := (i_1, i_2 - i_1, \ldots, i_k - i_1) \in \mathbb{Z}^k_{>0}$, we have $(\gamma'_1)^{j_1} \cdots (\gamma'_k)^{j_k} \sim \gamma_1^{i_1} \cdots \gamma_k^{i_k}$. Therefore, $\varrho((\gamma'_1)^{j_1} \cdots (\gamma'_k)^{j_k}) = 0$. In this case $j_1 + \cdots + j_k < i_1 + \cdots + i_k$. Next, we proceed to blow up the closure $\overline{D'_J}$ of D'_J and iterate the algorithm described above. This iterative process will terminate after a finite number of steps, resulting in a birational morphism $\mu : \overline{X}' \to \overline{X}$ that satisfies the properties described in Items (a) and (b). We repeat this algorithm of blowing-up for all other connected components Z of D_J with $|J| = k$ where ϱ does not have infinite monodromy at points in Z. By establishing and proving the induction, we complete the proof of the proposition. □

3.3. Construction of Shafarevich morphism (I). — We will construct the Shafarevich morphism for smooth quasi-projective varieties X associated to a constructible subsets of $M_{\mathrm{B}}(X, N)(\mathbb{C})$ defined over $\mathbb{Q}$ that is invariant under $\mathbb{R}^*$-action.

Theorem 3.20. — *Let X be a smooth quasi-projective variety. Let $\mathfrak{C}$ be a constructible subset of $M_{\mathrm{B}}(X, N)(\mathbb{C})$, defined over $\mathbb{Q}$, such that $\mathfrak{C}$ is invariant under $\mathbb{R}^*$-action. When X is non-compact, we make two additional assumptions:*

— *$\mathfrak{C}$ is closed;*
— *$\mathfrak{C}$ has infinite monodromy at infinity in the sense of Definition 3.14.*

Then there is a proper surjective holomorphic fibration $\mathrm{sh}_{\mathfrak{C}} : X \to \mathrm{Sh}_{\mathfrak{C}}(X)$ over a normal complex space *$\mathrm{Sh}_{\mathfrak{C}}(X)$ such that for any closed subvariety Z of X, $\mathrm{Sh}_{\mathfrak{C}}(Z)$ is a point if and only if $\varrho(\mathrm{Im}[\pi_1(Z) \to \pi_1(X)])$ is finite for any reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in \mathfrak{C}$. When X is compact, $\mathrm{Sh}_{\mathfrak{C}}(X)$ is projective.*

Proof. — We will divide the proof into two steps. The first step is dedicated to constructing $\mathrm{sh}_{\mathfrak{C}} : X \to \mathrm{Sh}_{\mathfrak{C}}(X)$. In the second step, we will prove the projectivity of $\mathrm{Sh}_{\mathfrak{C}}(X)$ when X is compact.

Step 1: constructing the Shafarevich morphism. By Proposition 3.12, there exist reduction representations $\{\sigma_i^{VHS} : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,m}$ that underlie $\mathbb{C}$-VHS such that, for a morphism $\iota : Z \to X$ from any quasi-projective *normal* variety Z with $s_{\mathfrak{C}} \circ \iota(Z)$ being a point, the following properties hold:

(a) For $\sigma := \oplus_{i=1}^m \sigma_i^{VHS}$, the image $\iota^*\sigma(\pi_1(Z))$ is discrete in $\prod_{i=1}^m \mathrm{GL}_N(\mathbb{C})$.
(b) For each reductive $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ with $[\tau] \in \mathfrak{C}(\mathbb{C})$, $\iota^*\tau$ is conjugate to some $\iota^*\sigma_i^{VHS}$. Moreover, for each σ_i^{VHS}, there exists some reductive representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ with $[\tau] \in \mathfrak{C}(\mathbb{C})$ such that $\iota^*\tau$ is conjugate to $\iota^*\sigma_i^{VHS}$.
(c) We have the following inclusion:

$$\cap_{[\varrho] \in \mathfrak{C}(\mathbb{C})} \ker \varrho \subset \ker \sigma_i^{VHS} \tag{3.7}$$

where ϱ varies in all reductive representations such that $[\varrho] \in \mathfrak{C}(\mathbb{C})$.

Define $H := \cap_\varrho \ker \varrho \cap \ker \sigma$, where $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representation such that $[\varrho] \in \mathfrak{C}(\mathbb{C})$. By (3.7) we have $H = \cap_\varrho \ker \varrho$, where $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representation such that $[\varrho] \in \mathfrak{C}(\mathbb{C})$. Denote by $\widetilde{X}_H := \widetilde{X}/H$. Let $\mathscr{D}$ be the period domain associated with the $\mathbb{C}$-VHS σ and let $p : \widetilde{X}_H \to \mathscr{D}$ be the period mapping. We define a holomorphic map

$$\Psi : \widetilde{X}_H \to S_{\mathfrak{C}} \times \mathscr{D}, \quad z \mapsto (s_{\mathfrak{C}} \circ \pi_H(z), p(z)) \tag{3.8}$$

where $\pi_H : \widetilde{X}_H \to X$ denotes the covering map and $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ is the reduction map defined in Definition 3.1.

Lemma 3.21. — *Each connected component of any fiber of Ψ is compact.*

Proof of Lemma 3.21. — It is equivalent to prove that for any $(t, o) \in S_{\mathfrak{C}} \times \mathscr{D}$, any connected component of $\Psi^{-1}(t, o)$ is compact. We fix any $t \in S_{\mathfrak{C}}$.

Step 1: we first assume that each irreducible component of $(s_{\mathfrak{C}})^{-1}(t)$ is normal. Let F be an irreducible component of $(s_{\mathfrak{C}})^{-1}(t)$. Then the natural morphism $\iota : F \to X$ is proper. By Item (a), $\Gamma := \sigma(\mathrm{Im}\,[\pi_1(F) \to \pi_1(X)])$ is a discrete subgroup of $\prod_{i=1}^m \mathrm{GL}_N(\mathbb{C})$.

Claim 3.22. — *The period mapping $F \to \mathscr{D}/\Gamma$ is proper.*

Proof. — Although F might be singular, we can still define its period mapping since it is normal. The definition is as follows: we begin by taking a resolution of singularities $\mu : E \to F$. Since F is normal, each fiber of μ is connected, and we have $\Gamma = \sigma(\mathrm{Im}[\pi_1(E) \to \pi_1(X)])$. It is worth noting that $\mathscr{D}/\Gamma$ exists as a complex normal space since Γ is discrete. Now, consider the period mapping $E \to \mathscr{D}/\Gamma$ for the $\mathbb{C}$-VHS induced $\mu^*\sigma$. This mapping then induces a holomorphic mapping $F \to \mathscr{D}/\Gamma$, which satisfies the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "\mu"] \arrow[d] & F \arrow[dl] \\ \mathscr{D}/\Gamma & \end{tikzcd}$$

The resulting holomorphic map $F \to \mathscr{D}/\Gamma$ is the period mapping for the $\mathbb{C}$-VHS on F induced by $\sigma|_{\pi_1(F)}$. To establish the properness of $F \to \mathscr{D}/\Gamma$, it suffices to prove that $E \to \mathscr{D}/\Gamma$ is proper. Let $\overline{X}$ be a smooth projective compactification such that $D := \overline{X}\backslash X$ is a simple normal crossing divisor. Given that $E \to X$ is a proper morphism, we can take a smooth projective compactification $\overline{E}$ of E such that

— the complement $D_E := \overline{E} \setminus E$ is a simple normal crossing divisor;
— there exists a morphism $j : \overline{E} \to \overline{X}$ such that $j^{-1}(D) = D_E$.

We aim to prove that $j^*\sigma : \pi_1(E) \to \prod_{i=1}^m \mathrm{GL}_N(\mathbb{C})$ has infinite monodromy at infinity.

Consider any holomorphic map $\gamma : \mathbb{D} \to \overline{E}$ such that $\gamma^{-1}(D_E) = \{0\}$. Then $(j \circ \gamma)^{-1}(D) = \{0\}$. As we assume that $\mathfrak{C}(\mathbb{C})$ has infinite monodromy at infinity, there exists a reductive representation $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\tau] \in \mathfrak{C}(\mathbb{C})$ and $(j \circ \gamma)^*\tau(\pi_1(\mathbb{D}^*))$ is infinite. Using Item (b), it follows that $j^*\tau$ and $j^*\sigma_i^{VHS}$ are conjugate to each other as E is smooth quasi-projective. As σ_i^{VHS} is a direct factor of σ, it follows that $(j \circ \gamma)^*\sigma(\pi_1(\mathbb{D}^*))$ is also infinite. Hence, we conclude that $j^*\sigma$ has infinite monodromy at infinity.

By a theorem of Griffiths (cf. [CMP17, Corollary 13.7.6]), we conclude that $E \to \mathscr{D}/\Gamma$ is proper. Therefore, $F \to \mathscr{D}/\Gamma$ is proper. □

Take any point $o \in \mathscr{D}$. Note that there is a real Lie group G_0 which acts holomorphically and transitively on $\mathscr{D}$. Let V be the compact subgroup that fixes o. Thus, we have $\mathscr{D} = G_0/V$. Now, let Z be any connected component of the fiber of $F \to \mathscr{D}/\Gamma$ over $[o]$. According to Claim 3.22, Z is guaranteed to be compact. We have that $\sigma(\mathrm{Im}\,[\pi_1(Z) \to \pi_1(X)]) \subset V \cap \Gamma$. Notably, V is compact, and Γ is discrete. As a result, it follows that $\sigma(\mathrm{Im}\,[\pi_1(Z) \to \pi_1(X)])$ is finite.

Claim 3.23. — $\mathrm{Im}\,[\pi_1(Z) \to \pi_1(X)] \cap H$ *is a finite index subgroup of* $\mathrm{Im}\,[\pi_1(Z) \to \pi_1(X)]$.

Proof. — By Item (b) and (3.7), we have

$$\ker\sigma \cap \mathrm{Im}\,[\pi_1(F)\to\pi_1(X)] = H\cap \mathrm{Im}\,[\pi_1(F)\to\pi_1(X)]. \tag{3.9}$$

Since $\sigma(\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)])$ is finite, $\ker\sigma\cap\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)]$ is a finite index subgroup of $\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)]$. The claim follows from (3.9). □

Pick any connected component Z_0 of $\pi_H^{-1}(Z)$. Note that $\mathrm{Aut}(Z_0/Z)=\frac{\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)]}{\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)]\cap H}$. According to Claim 3.23, $\mathrm{Aut}(Z_0/Z)$ is finite, implying that Z_0 is compact. Hence, $\pi_H^{-1}(Z)$ is a disjoint union of compact subvarieties of $\widetilde{X}_H$, each of which is a finite étale Galois cover of Z under π_H, with the Galois group $\frac{\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)]}{\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)]\cap H}$. If we denote by $\widetilde{F}$ a connected component of $\pi_H^{-1}(F)$, then each connected component of any fiber of $p|_{\widetilde{F}}:\widetilde{F}\to\mathscr{D}$ is a connected component of $\pi_H^{-1}(Z)$, which is compact. This can be illustrated by the following commutative diagram:

$$\begin{tikzcd} \widetilde{F} \arrow[r]\arrow[d] & F\arrow[d] \\ \mathscr{D}\arrow[r] & \mathscr{D}/\Gamma \end{tikzcd}$$

Since we have assumed that each irreducible component of $(s_{\mathfrak{C}})^{-1}(t)$ is normal, it follows that for any $o\in\mathscr{D}$, each connected component of $\Psi^{-1}(t,o)$ is compact.

Step 2: we prove the general case. In the general case, we consider an embedded resolution of singularities $\mu:Y\to X$ for the fiber $(s_{\mathfrak{C}})^{-1}(t)$ such that each irreducible component of $(s_{\mathfrak{C}}\circ\mu)^{-1}(t)$ is smooth. It is worth noting that $s_{\mathfrak{C}}\circ\mu:Y\to S_{\mathfrak{C}}$ coincides with the reduction map $s_{\mu^*\mathfrak{C}}:Y\to S_{\mu^*\mathfrak{C}}$ for $\mu^*\mathfrak{C}$. Let $\widetilde{Y}_H:=\widetilde{X}_H\times_X Y$, which is connected.

$$\begin{tikzcd} & \widetilde{Y}_H \arrow[r]\arrow[d, "\tilde{\mu}"] & Y\arrow[d, "\mu"] \\ S_{\mathfrak{C}}\times\mathscr{D} & \widetilde{X}_H \arrow[l, "\Psi"'] \arrow[r] & X \end{tikzcd}$$

We observe that $\tilde{\mu}$ is a proper holomorphic fibration. We define $H':=\cap_{\varrho}\ker\varrho\cap\ker\mu^*\sigma$, where $\varrho:\pi_1(Y)\to\mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representation such that $[\varrho]\in\mu^*\mathfrak{C}$. Since $(\mu_1)_*:\pi_1(Y)\to\pi_1(X)$ is an isomorphism, we have $(\mu_*)^{-1}(H)=H'$. Consequently, $\widetilde{Y}_H$ is the covering of Y corresponding to H', and thus $\mathrm{Aut}(\widetilde{Y}_H/Y)=H'\simeq H$. It is worth noting that $\mu^*\mathfrak{C}$ satisfying all the conditions required for $\mathfrak{C}$ as stated in Theorem 3.20, unless the $\mathbb{R}^*$-invariance is not obvious. However, we note that $\mu^*\mathfrak{C}$ is invariant by $\mathbb{R}^*$-action by Corollary 2.10. This enables us to work with $\mu^*\mathfrak{C}$ instead of $\mathfrak{C}$.

As a result, $\mu^*\sigma=\oplus_{i=1}^m\mu^*\sigma_i^{VHS}$ satisfies all the properties in Items (a) and (b) and eq. (3.7). Note that $\mu^*\sigma$ underlies a $\mathbb{C}$-VHS with the period mapping $p\circ\tilde{\mu}:\widetilde{Y}_H\to\mathscr{D}$. It follows that $\Psi\circ\tilde{\mu}:\widetilde{Y}_H\to S_{\mathfrak{C}}\times\mathscr{D}$ is defined in the same way as (3.8), determined by $\mu^*\mathfrak{C}$ and $\mu^*\sigma$.

Therefore, by Step 1, we can conclude that for any $o\in\mathscr{D}$, each connected component of $(\Psi\circ\tilde{\mu})^{-1}(t,o)$ is compact. Let Z be a connected component of $\Psi^{-1}(t,o)$. Then we claim that Z is compact. Indeed, $\tilde{\mu}^{-1}(Z)$ is closed and connected as each fiber of $\tilde{\mu}$ is connected. Therefore, $\tilde{\mu}^{-1}(Z)$ is contained in some connected component of $(\Psi\circ\tilde{\mu})^{-1}(t,o)$. So $\tilde{\mu}^{-1}(Z)$ is compact. As $\tilde{\mu}$ is proper and surjective, it follows that $Z=\tilde{\mu}(\tilde{\mu}^{-1}(Z))$ is compact. Lemma 3.21 is proved. □

As a resulf of Lemma 3.21 and Theorem 1.30, the set $\widetilde{S}_H$ of connected components of fibers of Ψ can be endowed with the structure of a complex normal space such that $\Psi=g\circ\mathrm{sh}_H$ where $\mathrm{sh}_H:\widetilde{X}_H\to\widetilde{S}_H$ is a proper holomorphic fibration and $g:\widetilde{S}_H\to\widetilde{S}_{\mathfrak{C}}\times\mathscr{D}$ is a holomorphic map. In Claim 3.31 below, we will prove that each fiber of g is discrete.

Claim 3.24. — sh_H *contracts every compact subvariety of* $\widetilde{X}_H$.

Proof. — Let $Z \subset \widetilde{X}_H$ be a compact irreducible subvariety. Then, $W := \pi_H(Z)$ is also a compact irreducible subvariety in X with $\dim Z = \dim W$. Hence $\mathrm{Im}\,[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(W^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(W^{\mathrm{norm}})$. Note that W can be endowed with an algebraic structure induced by X. As the natural map $Z \to W$ is finite, Z can be equipped with an algebraic structure such that the natural map $Z \to X$ is algebraic.

For any reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$ with $\varrho \in \mathfrak{C}(K)$ where K is a non archimedean local field, we have $\varrho(\mathrm{Im}\,[\pi_1(Z) \to \pi_1(X)]) \subset \varrho(\mathrm{Im}\,[\pi_1(\widetilde{X}_H) \to \pi_1(X)]) = \{1\}$. Hence, $\varrho(\mathrm{Im}\,[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X)])$ is finite which is thus bounded. By Lemma 3.2, W is contained in a fiber of $s_{\mathfrak{C}}$. Consider a desingularization Z' of Z and let $i : Z' \to X$ be the natural algebraic morphism. Note that $i^*\sigma(\pi_1(Z')) = \{1\}$. It follows that the variation of Hodge structure induced by $i^*\sigma$ is trivial. Therefore, $p(Z)$ is a point. Hence Z is contracted by Ψ. The claim follows. □

Lemma 3.25. — *There is an action of* $\mathrm{Aut}(\widetilde{X}_H/X) = \pi_1(X)/H$ *on* $\widetilde{S}_H$ *that is equivariant for the proper holomorphic fibration* $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$. *This action is analytic and properly discontinuous. Namely, for any point* y *of* $\widetilde{S}_H$, *there exists an open neighborhood* V_y *of* y *such that the set*

$$\{\gamma \in \pi_1(X)/H \mid \gamma.V_y \cap V_y \neq \varnothing\}$$

is finite.

Proof. — Take any $\gamma \in \pi_1(X)/H$. We can consider γ as an analytic automorphism of $\widetilde{X}_H$. According to Claim 3.24, $\mathrm{sh}_H \circ \gamma : \widetilde{X}_H \to \widetilde{S}_H$ contracts each fiber of the proper holomorphic fibration $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$. As a result, it induces a holomorphic map $\tilde{\gamma} : \widetilde{S}_H \to \widetilde{S}_H$ such that we have the following commutative diagram:

$$\begin{tikzcd} \widetilde{X}_H \arrow[r, "\gamma"] \arrow[d, "\mathrm{sh}_H"] & \widetilde{X}_H \arrow[d, "\mathrm{sh}_H"] \\ \widetilde{S}_H \arrow[r, "\tilde{\gamma}"] & \widetilde{S}_H \end{tikzcd}$$

Let us define the action of γ on $\widetilde{S}_H$ by $\tilde{\gamma}$. Then γ is an analytic automorphism and sh_H is $\pi_1(X)/H$-equivariant. It is evident that $\tilde{\gamma} : \widetilde{S}_H \to \widetilde{S}_H$ carries one fiber of sh_H to another fiber. Thus, we have shown that $\pi_1(X)/H$ acts on $\widetilde{S}_H$ analytically and equivariantly with respect to $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$. Now, we will prove that this action is properly discontinuous.

Take any $y \in \widetilde{S}_H$ and let $F := \mathrm{sh}_H^{-1}(y)$. Consider the subgroup $\mathcal{S}$ of $\pi_1(X)/H$ that fixes y, i.e.

$$\mathcal{S} := \{\gamma \in \pi_1(X)/H \mid \gamma \cdot F = F\}. \tag{3.10}$$

Since F is compact, $\mathcal{S}$ is finite.

Claim 3.26. — F *is a connected component of* $\pi_H^{-1}(\pi_H(F))$.

Proof of Claim 3.26. — Let $x \in \pi_H^{-1}(\pi_H(F))$. Then there exists $x_0 \in F$ such that $\pi_H(x) = \pi_H(x_0)$. Therefore, there exists $\gamma \in \pi_1(X)/H$ such that $\gamma.x_0 = x$. It follows that $\pi_H^{-1}(\pi_H(F)) = \cup_{\gamma \in \pi_1(X)/H}\gamma.F$. Since γ carries one fiber of Sh_H to another fiber, and the group $\pi_1(X)/H$ is finitely presented, it follows that $\cup_{\gamma \in \pi_1(X)/H}\gamma.F$ are countable union of fibers of Sh_H. It follows that F is a connected component of $\pi_H^{-1}(\pi_H(F))$. □

Claim 3.26 implies that $\pi_H : F \to \pi_H(F)$ is a finite étale cover. Denote by $Z := \pi_H(F)$ which is a connected Zariski closed subset of X. Then $\mathrm{Im}[\pi_1(F) \to \pi_1(Z)]$ is finite. As a consequence of [Hof09, Theorem 4.5], there is a connected open neighborhood U of Z such that $\pi_1(Z) \to \pi_1(U)$ is an isomorphism. Therefore, $\mathrm{Im}[\pi_1(U) \to \pi_1(W)] = \mathrm{Im}[\pi_1(F) \to$

$\pi_1(W)]$ is also finite. As a result, $\pi_H^{-1}(U)$ is a disjoint union of connected open sets $\{U_\alpha\}_{\alpha\in I}$ such that

(a) For each U_α, $\pi_H|_{U_\alpha}: U_\alpha \to U$ is a finite étale covering.
(b) Each U_α contains exactly one connected component of $\pi_H^{-1}(Z)$.

We may assume that $F \subset U_{\alpha_1}$ for some $\alpha_1 \in I$. By Item (b), for any $\gamma \in \pi_1(X,z)/H \simeq \mathrm{Aut}(\widetilde{X}_H/X)$, $\gamma\cdot U_{\alpha_1}\cap U_{\alpha_1}=\varnothing$ if and only if $\gamma\notin\mathcal{S}$.

Since sh_H is a proper holomorphic fibration, we can take a neighborhood V_y of y such that $\mathrm{sh}_H^{-1}(V_y)\subset U_{\alpha_1}$. Since $\gamma\cdot U_{\alpha_1}\cap U_{\alpha_1}=\varnothing$ if and only if $\gamma\notin\mathcal{S}$, it follows that $\gamma\cdot V_y\cap V_y=\varnothing$ if $\gamma\notin\mathcal{S}$. Since $\mathcal{S}$ is finite and y was chosen arbitrarily, we have shown that the action of $\pi_1(X)/H$ on $\widetilde{S}_H$ is properly discontinuous. Thus Lemma 3.25 is proven. □

Let $\nu:\pi_1(X)/H\to\mathrm{Aut}(\widetilde{S}_H)$ be action of $\pi_1(X)/H$ on $\widetilde{S}_H$ and let $\Gamma_0:=\nu(\pi_1(X)/H)$. By Lemma 3.25 and [Car60], we know that the quotient $\mathrm{Sh}_{\mathfrak{C}}(X):=\widetilde{S}_H/\Gamma_0$ is a complex normal space, and it is compact if X is compact. Moreover, since $\mathrm{sh}_H:\widetilde{X}\to\widetilde{S}_H$ is ν-equivariant, it induces a proper holomorphic fibration $\mathrm{sh}_{\mathfrak{C}}:X\to\mathrm{Sh}_{\mathfrak{C}}(X)$ from X to a complex normal space $\mathrm{Sh}_{\mathfrak{C}}(X)$.

$$
\begin{tikzcd}
& \widetilde{X}_H \arrow[r, "\pi_H"] \arrow[dl, "\Psi"'] \arrow[d, "\mathrm{sh}_H"] & X \arrow[d, "\mathrm{sh}_{\mathfrak{C}}"] \\
S_{\mathfrak{C}}\times\mathscr{D} \arrow[d] & \widetilde{S}_H \arrow[l, "g"'] \arrow[r, "\mu"] \arrow[dl, "\phi"] & \mathrm{Sh}_{\mathfrak{C}}(X) \\
\mathscr{D} & &
\end{tikzcd}
\tag{3.11}
$$

Claim 3.27. — *For any closed subvariety $Z\subset X$, $\mathrm{sh}_{\mathfrak{C}}(Z)$ is a point if and only if $\varrho(\mathrm{Im}[\pi_1(Z)\to\pi_1(X)])$ is finite for any reductive representation $\varrho:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ such that $[\varrho]\in\mathfrak{C}$.*

Proof. — *Proof of "⇐"*: Let $f:Y\to Z$ be a desingularization. Then for any non archimedean local field K and any reductive representation $\tau:\pi_1(X)\to\mathrm{GL}_N(K)$ with $[\tau]\in\mathfrak{C}(K)$, $f^*\tau(\pi_1(Y))$ is finite, and therefore bounded. Hence, $f(Y)$ is contained in some fiber F of $s_{\mathfrak{C}}$ by Lemma 3.2. Using Items (a) and (b), we have that $f^*\sigma(\pi_1(Y))$ is also finite. Therefore, Y is mapped to one point by the period mapping $Y\to\mathscr{D}/\Gamma$ of $f^*\sigma$. As a result, $\mathrm{sh}_{\mathfrak{C}}(Z)$ is a point by (3.11).

Proof of "⇒": Assume that $Z\subset X$ is a closed subvariety such that $\mathrm{sh}_{\mathfrak{C}}(Z)$ is a point. We observe from (3.11) that for any connected component Z' of $\pi_H^{-1}(Z)$, it is contracted by Ψ. By Lemma 3.21, Z' is contained in some compact subvariety of $\widetilde{X}_H$. Since Z' is closed, it is also compact. Therefore, the map $Z'\to Z$ induced by π_H is a finite étale cover.

Let $\varrho:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ be any reductive representation such that $[\varrho]\in\mathfrak{C}$. Note that $\varrho(\mathrm{Im}\,[\pi_1(Z')\to\pi_1(X)])$ is a finite index subgroup of $\varrho(\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)])$. Since $\varrho(\mathrm{Im}\,[\pi_1(Z')\to\pi_1(X)])=\{1\}$, it follows that $\varrho(\mathrm{Im}\,[\pi_1(Z)\to\pi_1(X)])$ is finite. The claim is proved. □

Therefore, we have constructed the desired proper holomorphic fibration $\mathrm{sh}_{\mathfrak{C}}:X\to\mathrm{Sh}_{\mathfrak{C}}(X)$. For the remaining part of the proof, we assume that X is compact and focus on proving the projectivity of $\mathrm{Sh}_{\mathfrak{C}}(X)$.

Step 2: projectivity of $\mathrm{Sh}_{\mathfrak{C}}(X)$ *if X is compact.*

Lemma 3.28. — *There exists a finite index normal subgroup N of $\pi_1(X)/H$ such that its action on $\widetilde{S}_H$ does not have fixed point.*

Proof of Lemma 3.28. — Let N_0 be a normal subgroup of $\pi_1(X)/H$. Consider the set of fixed points by N in $\widetilde{S}_H$ defined by

$$R_0 := \{y \in \widetilde{S}_H \mid \exists \gamma \in N_0 \text{ such that } \gamma \neq 1, \gamma.y = y.\}$$

Claim 3.29. — *R_0 is an analytic subset of $\widetilde{S}_H$, and it invariant under $\pi_1(X)/H$.*

Proof. — Take any $\gamma \in \pi_1(X)/H$ which is not the identity element. Consider the set of points in $\widetilde{S}_H$ fixed by γ defined by

$$F_\gamma := \{y \in \widetilde{S}_H \mid \gamma.y = y\}.$$

We claim that F_γ is an analytic subset. Indeed, if we define a holomorphic map

$$\begin{aligned} i_\gamma : \widetilde{S}_H &\to \widetilde{S}_H \times \widetilde{S}_H \\ y &\mapsto (y, \gamma.y), \end{aligned}$$

then $F_\gamma = i_\gamma^{-1}(\Delta)$, where Δ is the diagonal of $\widetilde{S}_H \times \widetilde{S}_H$. Hence, F_γ is an analytic subset of $\widetilde{S}_H$.

Observe that $R_0 = \cup_{\gamma \in N_0; \gamma \neq 1} F_\gamma$. Then we claim that R_0 is also an analytic subset of $\widetilde{S}_H$. Indeed, for any $y \in \widetilde{S}_H$, since the action of $\pi_1(X)/H$ on $\widetilde{S}_H$ is analytic and properly discontinuous, there exists an open neighborhood V_y of y such that $\mathcal{S}_y := \{\gamma \in N_0 \mid \gamma.V_y \cap V_y \neq \varnothing\}$ is finite. Therefore,

$$V_y \cap R_0 = V_y \cap (\cup_{\gamma \in N_0; \gamma \neq 1} F_\gamma) = V_y \cap (\cup_{\gamma \in \mathcal{S}_y; \gamma \neq 1} F_\gamma).$$

Hence, locally R_0 is a finite union of analytic subsets, that is also analytic subset. Therefore, R_0 is an analytic subset of $\widetilde{S}_H$. The first assertion is proved.

Take an arbitrary $y \in R_0$ and any $\gamma \in \pi_1(X)/H$ such that $\gamma.y \neq y$. Then there exists $\gamma_0 \in N_0$ such that $\gamma_0 \neq 1$ and $\gamma_0.y = y$. It follows that $(\gamma\gamma_0\gamma^{-1}).(\gamma.y) = \gamma.y$. Note that $\gamma\gamma_0\gamma^{-1} \neq 1$ and $\gamma\gamma_0\gamma^{-1} \in N_0$ as N_0 is normal. Hence $\gamma.y \in R_0$. Therefore, R_0 is invariant under $\pi_1(X)/H$. The claim is proved. □

Since $\mathrm{Sh}_{\mathfrak{C}}(X)$ is the quotient of $\widetilde{S}_H$ by $\pi_1(X)/H$, there exists an analytic subset $\mathcal{R}_0$ of $\mathrm{Sh}_{\mathfrak{C}}(X)$ such that $\mu^{-1}(\mathcal{R}_0) = R_0$, where $\mu : \widetilde{S}_H(X) \to \mathrm{Sh}_{\mathfrak{C}}(X)$ is the quotient map of $\widetilde{S}_H(X)$ by $\pi_1(X)/H$.

Claim 3.30. — *For any $y \in R_0$, there exists a finite index normal subgroup $N_1 \subset N_0$ such that for any $\gamma \in N_1$, $\gamma.y = y$ if and only if $\gamma = 1$.*

Proof. — Let $\mathcal{S}$ be the subgroup of N_0 that fixes y as defined in (3.10). It follows that $\mathcal{S}$ is a finite subgroup. By the definition of H, there exists a finite family of reductive representations $\{\varrho_i : X \to \mathrm{GL}_N(\mathbb{C})\}_{i=1,\ldots,\ell}$ with $[\varrho_i] \in \mathfrak{C}(\mathbb{C})$ such that $\cap_{i=1,\ldots,\ell} \ker \varrho_i \cap \mathcal{S} = \{1\}$. Considering the representation $\varrho_0 = \oplus_{i=1}^{\ell} \varrho_i : \pi_1(X)/H \to \prod_{i=1}^{\ell} \mathrm{GL}_N(\mathbb{C})$, the restriction $\varrho_0|_{\mathcal{S}} : \mathcal{S} \to \prod_{i=1}^{\ell} \mathrm{GL}_N(\mathbb{C})$ is injective. Since $\varrho_0(\pi_1(X)) = \varrho_0(\pi_1(X)/H)$ is finitely generated and linear, by Malcev's theorem, there exists a finite index subgroup Γ_1 of $\varrho_0(\pi_1(X)/H)$ such that $\Gamma_1 \cap \varrho_0(\mathcal{S}) = \{1\}$. Let $N_1 := \varrho_0^{-1}(\Gamma_1)$ which is a finite index subgroup of $\pi_1(X)/H$. Observe that $N_1 \cap \mathcal{S} = \{1\}$. Hence, the claim is proven. □

Let $N_0 := \pi_1(X)/H$, and let $R_0, \mathcal{R}_0$ be defined as above, induced by N_0. Now, let $y \in R_0$ be any point, and consider the finite index subgroup $N_1 \in \pi_1(X)/H$ as in Claim 3.30. It follows that the set of fixed points

$$R_1 := \{z \in \widetilde{S}_H \mid \exists \gamma \in N_1 \text{ such that } \gamma \neq 1, \gamma.z = z\}$$

does not contain y. By Claim 3.29, R_1 is invariant under $\pi_1(X)/H$ and thus there exists an analytic subset $\mathcal{R}_1$ of $\mathrm{Sh}_{\mathfrak{C}}(X)$ such that $\mu^{-1}(\mathcal{R}_1) = R_1$. It follows that $\mu(y) \notin \mathcal{R}_1$. Hence $\mathcal{R}_1 \subsetneq \mathcal{R}$.

We can iterate such procedure to find a decreasing sequence of finite index subgroups

$$\pi_1(X)/H = N_0 \supset N_1 \supset N_2 \supset \cdots$$

of $\pi_1(X)/H$ such that for the set of fixed points

$$R_k := \{x \in \widetilde{S}_H \mid \exists \gamma \in N_k \text{ such that } \gamma \neq 1, \gamma.x = x\}$$

it is invariant under $\pi_1(X)/H$ by Claim 3.29, and there exist analytic subsets $\mathcal{R}_k$ of $\mathrm{Sh}_{\mathfrak{C}}(X)$ such that $\mu^{-1}(\mathcal{R}_k) = R_k$ and $R_{k+1} \subsetneq R_k$. Since $\mathrm{Sh}_{\mathfrak{C}}(X)$ is compact, by the notherianity, $\mathcal{R}_k$ will stablise at some finite positive integer k_0. It is worth noting that $R_{k_0} = \varnothing$, or else we can still use the above algorithm to find $R_{k_0+1} \subsetneq R_{k_0}$. Therefore, we conclude that there exists a finite index subgroup $N := N_{k_0}$ of $\pi_1(X)/H$ which acts on $\widetilde{S}_H$ without fixed point. $\square$

Let $Y := \widetilde{X}_H/N$. Then $Y \to X$ is a finite Galois étale cover with $\mathrm{Aut}(\widetilde{X}_H/Y) = N$. Recall that we define $\nu : \pi_1(X)/H \to \mathrm{Aut}(\widetilde{S}_H)$ to be action of $\pi_1(X)/H$ on $\mathrm{Aut}(\widetilde{S}_H)$. Since $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$ is ν-equivariant, the group N gives rise to a proper holomorphic fibration $Y \to \widetilde{S}_H/\nu(N)$ over a complex normal space $\widetilde{S}_H/\nu(N)$. By Claim 3.22 and Lemma 3.28, $\nu(N)$ acts on $\widetilde{S}_H$ properly continuous and freely and thus the covering $\widetilde{S}_H \to \widetilde{S}_H/\nu(N)$ is étale.

Claim 3.31. — *Each fiber of $g : \widetilde{S}_H \to S_{\mathfrak{C}} \times \mathscr{D}$ is discrete.*

Proof. — Let $(t, o) \in S_{\mathfrak{C}} \times \mathscr{D}$ be arbitrary point and take any point $y \in g^{-1}((t, o))$. Then $Z := \mathrm{sh}_H^{-1}(y)$ is a connected component of the fiber $\Psi^{-1}((t, o))$, that is compact by Lemma 3.21. By Theorem 1.30, Z has an open neighborhood U such that $\Psi(U)$ is a locally closed analytic subvariety of $S_{\mathfrak{C}} \times \mathscr{D}$ and $\Psi|_U : U \to \Psi(U)$ is proper. Therefore, for the Stein factorization $U \to V \xrightarrow{\pi_V} \Psi(U)$ of $\Psi|_U$, $U \to V$ coincides with $\mathrm{sh}_H|_U : U \to \mathrm{sh}_H(U)$ and $\pi_V : V \to \Psi(U)$ is finite. Observe that V is an open neighborhood of y and $\pi_V : V \to \Psi(U)$ coincides with $g|_V : V \to S_{\mathfrak{C}} \times \mathscr{D}$. Therefore, the set $V \cap g^{-1}((t, o)) = V \cap (\pi_V)^{-1}(t, o)$ is finite. As a result, $g^{-1}((t, o))$ is discrete. The claim is proven. $\square$

In [Gri70], Griffiths discovered a so-called *canonical bundle $K_{\mathscr{D}}$ on the period domain* $\mathscr{D}$, which is invariant under G_0. Here G_0 is a real Lie group acting on $\mathscr{D}$ holomorphically and transitively. It is worth noting that $K_{\mathscr{D}}$ is endowed with a G_0-invariant smooth metric $h_{\mathscr{D}}$ whose curvature is positive-definite in the horizontal direction. The period mapping $p : \widetilde{X}_H \to \mathscr{D}$ induces a holomorphic map $\phi : \widetilde{S}_H \to \mathscr{D}$ which is horizontal. We note that ϕ is ν-equivariant. As a result, $\phi^* K_{\mathscr{D}}$ descends to a line bundle on the quotient $W := \widetilde{S}_H/\nu(N)$, denoted by L_G. The smooth metric $h_{\mathscr{D}}$ induces a smooth metric h_G on L_G whose curvature form is denoted by T. Let $x \in \widetilde{S}_H$ be a smooth point of $\widetilde{S}_H$ and let $v \in T_{\widetilde{S}_H, x}$. Then $-iT(v, \bar{v}) > 0$ if $d\phi(v) \neq 0$.

Claim 3.32. — $\mathrm{Sh}_{\mathfrak{C}}(X)$ *is a projective normal variety.*

Proof. — Note that $S_{\mathfrak{C}}$ is a projective normal variety. We take an ample line bundle L over $S_{\mathfrak{C}}$. Recall that there is a line bundle L_{G} on W equipped with a smooth metric h_{G} such that its curvature form is T. Denote by $f : W \to S_{\mathfrak{C}}$ the natural morphism induced by $g : \widetilde{S}_H \to S_{\mathfrak{C}} \times \mathscr{D}$. Let $\mu : W' \to W$ be a resolution of singularities of W.

$$\begin{tikzcd}
 & \widetilde{X}_H \arrow[r, "\pi_H"] \arrow[dl, "\Psi"'] \arrow[d, "\mathrm{sh}_H"] & Y \arrow[d] & \\
S_{\mathfrak{C}} \times \mathscr{D} \arrow[d] & \widetilde{S}_H \arrow[l, "g"'] \arrow[r] \arrow[dl, "\phi"] & W \arrow[d, "f"] & W' \arrow[l, "\mu"'] \\
\mathscr{D} & & S_{\mathfrak{C}} &
\end{tikzcd}$$

We take a smooth metric h on L such that its curvature form $i\Theta_h(L)$ is Kähler. As shown in Claim 3.31, the map $g : \widetilde{S}_H \to S_{\mathfrak{C}} \times \mathscr{D}$ is discrete. Therefore, g is an immersion at general points of $\widetilde{S}_H$. Thus, for the line bundle $\mu^*(L_{\mathrm{G}} \otimes f^*L)$ equipped with the smooth metric $\mu^*(h \otimes f^*h_{\mathrm{G}})$, its curvature form is strictly positive at some points of W', . By

Demailly's holomorphic Morse inequality or Siu's solution for the Grauert-Riemenschneider conjecture, $\mu^*(L_{\mathrm{G}} \otimes f^*L)$ is a big line bundle and thus W' is a Moishezon manifold. Hence W is a Moishezon variety.

Moreover, we can verify that for irreducible positive-dimensional closed subvariety Z of W, there exists a smooth point x in Z such that it has a neighborhood Ω that can be lifted to the étale covering $\widetilde{S}_H$ of W, and $g|_\Omega : \Omega \to S_{\mathfrak{C}} \times \mathscr{D}$ is an immersion. It follows that $(if^*\Theta_h(L) + T)|_\Omega$ is strictly positive. Note that

$$(L_G \otimes f^*L)^{\dim Z} \cdot [Z] = \int_{Z^{\mathrm{reg}}} (if^*\Theta_h(L) + T)^{\dim Z} > 0.$$

By the Nakai-Moishezon criterion for Moishezon varieties (cf. [Kol90, Theorem 3.11]), $L_G \otimes f^*L$ is ample , implying that W is projective. Recall that the compact complex normal space $\mathrm{Sh}_{\mathfrak{C}}(X) := \widetilde{S}_H/\Gamma_0$ is a quotient of $W = \widetilde{S}_H/\nu(N)$ by the finite group $\Gamma_0/\nu(N)$. Therefore, $\mathrm{Sh}_{\mathfrak{C}}(X)$ is also projective. The claim is proved. □

We accomplish the proof of the theorem. □

Remark 3.33. — We remark that Lemma 3.25 is claimed without a proof in [Eys04, p. 524] and [Bru23, Proof of Theorem 10]. It appears to us that the proof of Lemma 3.25 is not straightforward.

It is worth noting that Lemma 3.28 is implicitly used in [Eys04, Proposition 5.3.10]. In that proof, the criterion for Stein spaces (cf. Proposition 1.14) is employed, assuming Lemma 3.28. The proof of Lemma 3.28 is non-trivial, particularly considering that $\pi_1(X)/H$ may not be residually finite. Given its significance in the proofs of Theorems B and C, we provide a complete proof. It is noteworthy that our proof of Lemma 3.28 is valid only for the compact case, and extending it to the quasi-projective case is not straightforward due to the reliance on the compactness of $\mathrm{Sh}_{\mathfrak{C}}(X)$.

3.4. Construction of Shafarevich morphism (II). — In the previous subsection, we established the existence of the Shafarevich morphism associated with a constructible subset of $M_{\mathrm{B}}(X, N)(\mathbb{C})$ defined over $\mathbb{Q}$ that are invariant under $\mathbb{R}^*$-action. In this section, we focus on proving an existence theorem for the Shafarevich morphism associated with a single reductive representation, based on Theorem 3.20. Initially, we assume that the representation has infinite monodromy at infinity. However, we will subsequently employ Proposition 3.17 to remove this assumption and establish the more general result.

Proposition 3.34. — *Let X be a quasi-projective normal variety. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. Assume that ϱ has infinite monodromy at infinity if X is non-compact. Then there exists a proper surjective holomorphic fibration $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ onto a complex normal space $\mathrm{Sh}_\varrho(X)$ such that for any closed subvariety $Z \subset X$, $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite if and only if $\mathrm{sh}_\varrho(Z)$ is a point. If X is compact, then $\mathrm{Sh}_\varrho(X)$ is projective.*

We first prove the following crucial result.

Proposition 3.35. — *Let X be a smooth quasi-projective variety. Let $f : Z \to X$ be a* proper *morphism from a* smooth *quasi-projective variety Z. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. Define $M := j_Z^{-1}\{1\}$, where 1 stands for the trivial representation, and $j_Z : M_{\mathrm{B}}(X, N) \to M_{\mathrm{B}}(Z, N)$ is the natural morphism of $\mathbb{Q}$-scheme. Then M is a closed subscheme of $M_{\mathrm{B}}(X, N)$ defined over $\mathbb{Q}$ such that $M(\mathbb{C})$ is invariant under $\mathbb{C}^*$-action.*

Proof. — We take a smooth projective compactification $\overline{X}$ (resp. $\overline{Z}$) of X (resp. Z) such that $D := \overline{X} \backslash X$ (resp. $D_Z := \overline{Z} \backslash Z$) is a simple normal crossing divisor and f extends to a morphism $\bar{f} : \overline{X} \to \overline{Z}$. Note that the morphism j_Z is a $\mathbb{Q}$-morphism between affine schemes of finite type $M_{\mathrm{B}}(X, N)$ and $M_{\mathrm{B}}(Z, N)$ defined over $\mathbb{Q}$. M is thus a closed subscheme of $M_{\mathrm{B}}(X, N)$ defined over $\mathbb{Q}$. Let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation such that $[\varrho] \in M(\mathbb{C})$. By Theorem 1.6, there is a tame pure imaginary harmonic bundle (E, θ, h) on X such that ϱ is the monodromy representation of $\nabla_h + \theta + \theta_h^\dagger$. By definition,

$f^*\varrho$ is a trivial representation. Therefore, $f^*\varrho$ corresponds to a trivial harmonic bundle $(\oplus^N \mathcal{O}_Z, 0, h_0)$ where h_0 is the canonical metric for the trivial vector bundle $\oplus^N \mathcal{O}_Z$ with zero curvature. By the unicity theorem in [Moc06, Theorem 1.4], $(\oplus^N \mathcal{O}_Z, 0, h_0)$ coincides with $(f^*E, f^*\theta, f^*h)$ with some obvious ambiguity of h_0. Therefore, $f^*E = \oplus^N \mathcal{O}_Z$ and $f^*\theta = 0$. In particular, the regular filtered Higgs bundle $(\tilde{E}_*, \tilde{\theta})$ on $(\overline{Z}, D_Z)$ induced by the prolongation of $(f^*E, f^*\theta, f^*h)$ using norm growth defined in § 2.1 is trivial; namely we have ${}_{\boldsymbol{a}}\tilde{E} = \mathcal{O}_{\overline{Z}}^N \otimes \mathcal{O}_{\overline{Z}}(\sum_{i=1}^{\ell} a_i D_i')$ for any $\boldsymbol{a} = (a_1, \ldots, a_m) \in \mathbb{R}^\ell$ and $\tilde{\theta} = 0$. Here we write $D_Z = \sum_{i=1}^{\ell} D_i'$.

Let $(\boldsymbol{E}_*, \theta)$ be the induced regular filtered Higgs bundle on $(\overline{X}, D)$ by (E, θ, h) defined in § 2.1. According to §§ 2.2 and 2.3 we can define the pullback $(f^*\boldsymbol{E}_*, f^*\theta)$, which also forms a regular filtered Higgs bundle on $(\overline{Z}, D_Z)$ with trivial characteristic numbers. By virtue of Proposition 2.5, we deduce that $(f^*\boldsymbol{E}_*, f^*\theta) = (\tilde{E}_*, \tilde{\theta})$. Consequently, it follows that $(f^*\boldsymbol{E}_*, f^*\theta)$ is trivial. Hence $(f^*\boldsymbol{E}_*, tf^*\theta)$ is trivial for any $t \in \mathbb{C}^*$.

Fix some ample line bundle L on $\overline{Z}$. It is worth noting that for any $t \in \mathbb{C}^*$, $(\boldsymbol{E}_*, t\theta)$ is μ_L-polystable with trivial characteristic numbers. By [Moc06, Theorem 9.4], there is a pluriharmonic metric h_t for $(E, t\theta)$ adapted to the parabolic structures of $(\boldsymbol{E}_*, t\theta)$. By Proposition 2.5 once again, the regular filtered Higgs bundle $(f^*\boldsymbol{E}_*, tf^*\theta)$ is the prolongation of the tame harmonic bundle $(f^*E, tf^*\theta, f^*h_t)$ using norm growth defined in § 2.1. Since $(f^*\boldsymbol{E}_*, tf^*\theta)$ is trivial for any $t \in \mathbb{C}^*$, by the unicity theorem in [Moc06, Theorem 1.4] once again, it follows that $(\oplus^N \mathcal{O}_Z, 0, h_0)$ coincides with $(f^*E, tf^*\theta, f^*h_t)$ with some obvious ambiguity of h_0. Recall that in § 2.4, ϱ_t is defined to be the monodromy representation of the flat connection $\nabla_{h_t} + t\theta + \bar{t}\theta_{h_t}^\dagger$. It follows that $f^*\varrho_t$ is the monodromy representation of the flat connection $f^*(\nabla_{h_t} + t\theta + \bar{t}\theta_{h_t}^\dagger)$. Therefore, $f^*\varrho_t$ is a trivial representation.

However, it is worth noting that ϱ_t might not be reductive as $(E, t\theta, h_t)$ might not be pure imaginary. Let ϱ_t^{ss} be the semisimplification of ϱ_t. Then $[\varrho_t] = [\varrho_t^{ss}]$. Since $f^*\varrho_t$ is a trivial representation, then $f^*\varrho_t^{ss}$ is trivial. The proposition is proved. □

Remark 3.36. — It is important to note that, unlike the projective case, the proof of Proposition 3.35 becomes considerably non-trivial when X is quasi-projective. This complexity arises from the utilization of the functoriality of pullback of regular filtered Higgs bundles, which is established in Proposition 2.5. Lemma 3.35 plays a crucial role in the proof of Proposition 3.34 as it allows us to remove the condition of $\mathbb{R}^*$-invariance in Theorem 3.20. However, we remark that Proposition 3.35 is claimed without a proof in the proof of [Bru23, Lemma 9.3].

Proof of Proposition 3.34. — *Step 1: We assume that X is smooth.* Let $f : Z \to X$ be a *proper* morphism from a *smooth* quasi-projective variety Z. Then $j_Z : M_{\mathrm{B}}(X, N) \to M_{\mathrm{B}}(Z, N)$ is a morphism of $\mathbb{Q}$-scheme. Define

$$\mathfrak{C} := \bigcap_{\{f: Z \to X \mid f^*\varrho = 1\}} j_Z^{-1}\{1\}, \tag{3.12}$$

where 1 stands for the trivial representation, and $f : Z \to X$ ranges over all proper morphisms from smooth quasi-projective varieties Z to X. Then $\mathfrak{C}$ is a zariski closed subset defined over $\mathbb{Q}$, and by Proposition 3.35, $\mathfrak{C}(\mathbb{C})$ is invariant under $\mathbb{C}^*$-action. Note that $[\varrho] \in \mathfrak{C}(\mathbb{C})$. As we assume that ϱ has infinite monodromy at infinity, conditions in Theorem 3.20 are fulfilled. Therefore, we apply Theorem 3.20 to conclude that the Shafarevich morphism $\mathrm{sh}_{\mathfrak{C}} : X \to \mathrm{Sh}_{\mathfrak{C}}(X)$ exists. It is a proper holomorphic fibration over a complex normal space.

Claim 3.37. — *For any proper morphism $f : Z \to X$ from a smooth quasi-projective variety Z, $f^*\varrho(\pi_1(Z))$ is finite if and only if $\mathrm{sh}_{\mathfrak{C}}(Z)$ is a point.*

Proof. — *Proof of "⇐"*: this follows from the fact that $[\varrho] \in \mathfrak{C}(\mathbb{C})$ and Theorem 3.20.

Proof of "⇒": we take a finite étale cover $Y \to Z$ such that $f^*\varrho(\mathrm{Im}[\pi_1(Y) \to \pi_1(Z)])$ is trivial. Denote by $g : Y \to X$ the composition of f with $Y \to Z$. Then g is proper and

$g^*\varrho = 1$. Let $\tau : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be any reductive representation such that $[\tau] \in \mathfrak{C}(\mathbb{C})$. Then $g^*\tau = 1$ by (3.12). It follows that $f^*\tau(\pi_1(Z))$ is finite. The lemma is proved. □

Let $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ be $\mathrm{sh}_{\mathfrak{C}} : X \to \mathrm{Sh}_{\mathfrak{C}}(X)$. The proposition is proved if X is smooth.

Step 2: We does not assume that X is smooth. We take a desingularization $\mu : Y \to X$. Then $\mu^*\varrho : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ is also a reductive representation. By Lemma 3.16 it also has infinite monodromy at infinity when X is non-compact. Based on the first step, the Shafarevich morphism $\mathrm{sh}_{\mu^*\varrho} : Y \to \mathrm{Sh}_{\mu^*\varrho}(Y)$ exists, which is a surjective proper holomorphic fibration. Let Z be an irreducible component of a fiber of μ. Then $\mu^*(\pi_1(Z)) = \{1\}$. It follows that $\mathrm{sh}_{\mu^*\varrho}(Z)$ is a point. Note that each fiber of μ is connected as X is normal. It follows that each fiber of μ is contracted to a point by $\mathrm{sh}_{\mu^*\varrho}$. Therefore, there exists a dominant holomorphic map $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_{\mu^*\varrho}(Y)$ with connected general fibers such that we have the following commutative diagram:

$$\begin{tikzcd} Y \arrow[d, "\mu"'] \arrow[dr, "\mathrm{sh}_{\mu^*\varrho}"] & \\ X \arrow[r, "\mathrm{sh}_\varrho"] & \mathrm{Sh}_{\mu^*\varrho}(Y) \end{tikzcd} \tag{3.13}$$

Claim 3.38. — *For any closed subvariety $Z \subset X$, $\mathrm{sh}_\varrho(Z)$ is a point if and only if $\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)]$ is finite.*

Proof. — Let us choose an irreducible component W of $\mu^{-1}(Z)$ which is surjective onto Z. Since $\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(Z^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(Z^{\mathrm{norm}})$, and $\mu_* : \pi_1(Y) \to \pi_1(X)$ is surjective, it follows that $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite if and only if $\mu^*\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(Y)])$ is finite.

Proof of $\Rightarrow$: Note that $\mathrm{sh}_{\mu^*\varrho}(W)$ is a point and thus $\mu^*\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(Y)])$ is finite. Hence $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite.

Proof of $\Leftarrow$: Note that $\mu^*\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(Y)])$ is finite. Therefore, $\mathrm{sh}_{\mu^*\varrho}(W)$ is a point and thus $\mathrm{sh}_\varrho(Z)$ is a point by (3.13). □

Let us write $\mathrm{Sh}_\varrho(X) := \mathrm{Sh}_{\mu^*\varrho}(Y)$. Then $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ is the Shafarevich morphism associated with $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$. □

The condition in Proposition 3.34 that ϱ has infinite monodromy at infinity poses significant practical limitations for further applications. However, we can overcome this this drawback by utilizing Proposition 3.17, which allows us to eliminate this requirement.

Theorem 3.39. — *Let X be a non-compact, quasi-projective normal varieties, and let $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. Then there exists a dominant holomorphic map $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ to a complex normal space $\mathrm{Sh}_\varrho(X)$ whose general fibers are connected such that for any closed subvariety $Z \subset X$, $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite if and only if $\mathrm{sh}_\varrho(Z)$ is a point.*

Proof. — By Step 2 of the proof of Proposition 3.34, it suffices to prove the theorem for X being a smooth variety. Therefore, we can replace X by its desingularization and replace ϱ by its pullback over this smooth model. Since $\varrho(\pi_1(X))$ is residually finite by Malcev's theorem, we can find a finite étale cover $\nu_0 : \widehat{X} \to X$ such that $\nu_0^*\varrho$ is torsion free.

Claim 3.40. — *There are partial compactifications X' (resp. $\widehat{X}'$) of X (resp. $\widehat{X}$) such that*

— *$\widehat{X}'$ and X' are quasi-projective normal varieties;*
— *$\nu_0 : \widehat{X} \to X$ extends to a finite morphism $\nu' : \widehat{X}' \to X'$;*
— *$\pi^*\varrho$ extends to a reductive representation $\varrho' : \pi_1(\widehat{X}') \to \mathrm{GL}_N(\mathbb{C})$ that has infinite monodromy at infinity.*

Proof. — Let $\overline{X}$ be a smooth projective compactification of X. Then there exists a smooth projective variety $\overline{\widehat{X}}_1$ that compactifies $\widehat{X}$, and a surjective generically finite morphism $\nu_1 : \overline{\widehat{X}}_1 \to \overline{X}$ that extends ν_0.

By utilizing Proposition 3.17, after replacing $\overline{\widehat{X}}_1$ by some birational modification, there exists a simple normal crossing divisor $D \subset \overline{\widehat{X}}_1$ such that we have $\widehat{X}_1 := \overline{\widehat{X}}_1 \backslash D \supset \widehat{X}$ and $\nu^*\varrho$ extends to a representation $\varrho_1 : \pi_1(\widehat{X}_1) \to \mathrm{GL}_N(\mathbb{C})$ that has infinite monodoromy at infinity.

The morphism $\nu_1 : \overline{\widehat{X}}_1 \to \overline{X}$ is not necessarily finite, but the restriction $\nu_1|_{\widehat{X}} : \widehat{X} \to X$ is a finite étale cover. By applying Hironaka-Raynaud-Gruson's flattening theorem, we can find a birational morphism $\overline{X}_1 \to \overline{X}$ that is isomorphic over X such that for the base change $\overline{\widehat{X}}_1 \times_{\overline{X}} \overline{X}_1 \to \overline{X}_1$, the main component denoted as $(\overline{\widehat{X}}_1 \times_{\overline{X}} \overline{X}_1)_{\mathrm{main}}$ which dominates $\overline{X}_1$, is flat over $\overline{X}_1$. Let $\overline{\widehat{X}}$ be the normalization of $(\overline{\widehat{X}}_1 \times_{\overline{X}} \overline{X}_1)_{\mathrm{main}}$.

$$
\begin{tikzcd}
\overline{\widehat{X}} \arrow[r] \arrow[rr, bend left, "\mu"] \arrow[dr, "\nu"'] & (\overline{\widehat{X}}_1 \times_{\overline{X}} \overline{X}_1)_{\mathrm{main}} \arrow[r] \arrow[d] & \overline{\widehat{X}}_1 \arrow[d, "\nu_1"] \\
 & \overline{X}_1 \arrow[r, "\mu_0"] & \overline{X}
\end{tikzcd}
$$

Then ν is a finite morphism. Let's define $D' := \mu^{-1}(D)$. Now, consider the pullback $\mu^*\varrho_1 : \pi_1(\overline{\widehat{X}} \backslash D') \to \mathrm{GL}_N(\mathbb{C})$. By Lemma 3.15, we observe that it has infinite monodromy at infinity. Consequently, $(\mu_0 \circ \nu)^*\varrho$ has infinite monodromy at each point of D' in the sense of Lemma 3.16. Next, we assert that $\nu^{-1}(\nu(D')) = D'$.

To establish the claim, we need to show that $\mu_0^*\varrho$ has infinite monodromy at each point of $\nu(D')$. Assume, for the sake of contradiction, that there exists $q \in \nu(D')$ such that $\mu_0^*\varrho$ does not have infinite monodromy at q. Let $p' \in D'$ be such that $\nu(p') = q$. Then there exists a holomorphic map $f : \mathbb{D} \to \overline{X}_1$ such that

— $f(\mathbb{D}^*) \subset X$ and $f(0) = q$;
— $f^*(\mu_0^*\varrho)(\pi_1(\mathbb{D}^*)) = \{1\}$;
— $f_*(\pi_1(\mathbb{D}^*)) \subset \mathrm{Im}[\pi_1(\widehat{X}) \to \pi_1(X)]$.

Since $\widehat{X} \to X$ is a finite étale cover, there exists a holomorphic map $\hat{f} : \mathbb{D} \to \overline{\widehat{X}}$ such that $\nu \circ \hat{f} = f$, $\hat{f}(\mathbb{D}^*) \subset \widehat{X}$ and $\hat{f}(0) = p'$. Therefore, we have $\hat{f}^*((\mu_0 \circ \nu)^*\varrho)(\pi_1(\mathbb{D}^*)) = \{1\}$. However, this contradicts the fact that $(\mu_0 \circ \nu)^*\varrho$ has infinite monodromy at each point of D'. We conclude that $\mu_0^*\varrho$ has infinite monodromy at each point of $\nu(D')$. Then $\nu^*(\mu_0^*\varrho)$ has infinite monodromy at each point of $\nu^{-1}(\nu(D'))$. We observe that $\mu^*\varrho_1$ is the extension of $\nu^*(\mu_0^*\varrho) : \pi_1(\widehat{X}) \to \mathrm{GL}_N(\mathbb{C})$ over $\overline{\widehat{X}} \backslash D'$. Consequently, we have $\nu^{-1}(\nu(D')) = D'$. Hence, $\nu|_{\overline{\widehat{X}} \backslash D'} : \overline{\widehat{X}} \backslash D' \to \overline{X}_1 \backslash \nu(D')$ is a finite morphism.

The claim follows by denoting $\widehat{X}' := \overline{\widehat{X}} \backslash D'$, $X' := \overline{X}_1 \backslash \nu(D')$ and $\nu' := \nu|_{\widehat{X}'}$. □

We proceed by finding a finite morphism $h : Y' \to \widehat{X}'$ from a normal quasi-projective variety Y' such that the composition $f : Y' \to X'$ of $\widehat{X}' \to X'$ and $Y' \to \widehat{X}'$ is a Galois cover with Galois group G. By Claim 3.40 and Lemma 3.15, $h^*\varrho' : \pi_1(Y') \to \mathrm{GL}_N(\mathbb{C})$ also has infinite monodromy at infinity. Consequently, we can apply Proposition 3.34 to deduce the existence of a proper holomorphic fibration $\mathrm{sh}_{h^*\varrho'} : Y' \to \mathrm{Sh}_{h^*\varrho'}(Y')$ such that for any closed subvariety Z of Y', $\mathrm{sh}_{h^*\varrho'}(Z)$ is a point if and only if $h^*\varrho'(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(Y')])$ is finite.

Claim 3.41. — *The Galois group G acts analytically on $\mathrm{Sh}_{h^*\varrho'}(Y')$ such that $\mathrm{sh}_{h^*\varrho'}$ is G-equivariant.*

Proof. — Take any $y \in \mathrm{Sh}_{h^*\varrho'}(Y')$ and any $g \in G$. Since $\mathrm{sh}_{h^*\varrho'}$ is surjective and proper, the fiber $\mathrm{sh}_{h^*\varrho'}^{-1}(y)$ is thus non-empty and compact. Let Z be an irreducible component of the fiber $\mathrm{sh}_{h^*\varrho'}^{-1}(y)$. Then $h^*\varrho'(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(Y')])$ is finite, implying that $h^*\varrho'(\mathrm{Im}[\pi_1((g.Z)^{\mathrm{norm}}) \to \pi_1(Y')])$ is also finite. Consequently, there exists a point $y' \in \mathrm{Sh}_{h^*\varrho'}(Y')$ such that $\mathrm{sh}_{h^*\varrho'}(g.Z) = y'$. Since each fiber of $\mathrm{sh}_{h^*\varrho'}$ is connected, for any other irreducible component Z' of $\mathrm{sh}_{h^*\varrho'}^{-1}(y)$, we have $\mathrm{sh}_{h^*\varrho'}(g.Z') = y'$. Consequently, it follows that g maps each fiber of $\mathrm{sh}_{h^*\varrho'}$ to another fiber.

We consider g as an analytic automorphism of Y'. For the holomorphic map $\mathrm{sh}_{h^*\varrho'} \circ g : Y' \to \mathrm{Sh}_{h^*\varrho'}(Y')$, since it contracts each fiber of $\mathrm{sh}_{h^*\varrho'} : Y' \to \mathrm{Sh}_{h^*\varrho'}(Y')$, it induces a holomorphic map $\tilde{g} : \mathrm{Sh}_{h^*\varrho'}(Y') \to \mathrm{Sh}_{h^*\varrho'}(Y')$ such that we have the following commutative diagram:

$$\begin{tikzcd}
Y' \arrow[r, "g"] \arrow[d, "\mathrm{sh}_{h^*\varrho'}"] & Y' \arrow[d, "\mathrm{sh}_{h^*\varrho'}"] \\
\mathrm{Sh}_{h^*\varrho'}(Y') \arrow[r, "\tilde{g}"] & \mathrm{Sh}_{h^*\varrho'}(Y')
\end{tikzcd} \tag{3.14}$$

Let us define the holomorphic map $\tilde{g} : \mathrm{Sh}_{h^*\varrho'}(Y') \to \mathrm{Sh}_{h^*\varrho'}(Y')$ to be the action of $g \in G$ on $\mathrm{Sh}_{h^*\varrho'}(Y')$. Based on (3.14), it is clear that $\mathrm{sh}_{h^*\varrho'}$ is G-equivariant. Therefore, the claim is proven. □

Note that $X' := Y'/G$. The quotient of $\mathrm{Sh}_{h^*\varrho'}(Y')$ by G, resulting in a complex normal space denoted by Q (cf. [Car60]). Then $\mathrm{sh}_{h^*\varrho'}$ induces a proper holomorphic fibration $c' : X' \to Q$. Consider the restriction $c := c'|_X$.

$$\begin{tikzcd}
Y \arrow[r, "f_0"] \arrow[d, hook] & X \arrow[d, hook] \arrow[dd, bend left, "c"] \\
Y' \arrow[r, "f"] \arrow[d, "\mathrm{sh}_{h^*\varrho'}"] & X' \arrow[d, "c'"] \\
\mathrm{Sh}_{h^*\varrho'}(Y') \arrow[r] & Q
\end{tikzcd}$$

Claim 3.42. — *For any closed subvariety Z of X, $c(Z)$ is a point if and only if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(Y')])$ is finite.*

Proof. — Let $Y := f^{-1}(X)$ and $f_0 := f|_Y$. Note that $f_0 : Y \to X$ is a Galois cover with Galois group G. We have $h^*\varrho'|_{\pi_1(Y)} = f_0^*\varrho$. Now, consider any closed subvariety Z of X. There exists an irreducible closed subvariety W of Y such that $f_0(W) = Z$. Let $\overline{W}$ be the closure of W in Y', which is an irreducible closed subvariety of Y'.

Observe that $c(Z)$ is a point if and only if $\mathrm{sh}_{h^*\varrho'}(\overline{W})$ is a point, which is equivalent to $h^*\varrho'(\mathrm{Im}[\pi_1(\overline{W}^{\mathrm{norm}}) \to \pi_1(Y')])$ being finite. Furthermore, this is equivalent to $f_0^*\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(Y)])$ being finite since $h^*\varrho'|_{\pi_1(Y)} = f_0^*\varrho$. Since $\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(Z^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(Z^{\mathrm{norm}})$, the above condition is equivalent to $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ being finite. □

Let $f := \mathrm{sh}_\varrho$ and $Q := \mathrm{Sh}_\varrho(X)$. This concludes our construction of the Shafarevich morphism of ϱ. Therefore, our theorem is proven. □

Corollary 3.43. — *Let X be a quasi-projective normal variety. Let Σ be a (non-empty) set of reductive representations $\varrho : \pi_1(X) \to \mathrm{GL}_{N_\varrho}(\mathbb{C})$. If X is non-compact, we assume additionally that each ϱ has infinite monodromy at infinity. Then there is a proper surjective holomorphic fibration $\mathrm{sh}_\Sigma : X \to \mathrm{Sh}_\Sigma(X)$ onto a complex normal space such that for closed subvariety $Z \subset X$, $\mathrm{sh}_\Sigma(Z)$ is a point if and only if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite for every $\varrho \in \Sigma$. Moreover, $\mathrm{Sh}_\Sigma(X)$ is a projective normal variety if X is compact.*

Proof. — By Proposition 3.34, for each $\varrho \in \Sigma$, there exists a surjective proper holomorphic fibration $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ onto a complex normal space $\mathrm{Sh}_\varrho(X)$. By [Car60], there exists a surjective proper holomorphic fibration $\mathrm{sh}_\Sigma : X \to \mathrm{Sh}_\Sigma(X)$ onto a complex normal space $\mathrm{Sh}_\Sigma(X)$ and holomorphic maps $e_\varrho : \mathrm{Sh}_\Sigma(X) \to \mathrm{Sh}_\varrho(X)$ such that

(a) $\mathrm{sh}_\varrho = e_\varrho \circ \mathrm{sh}_\Sigma$;
(b) for any $y \in \mathrm{Sh}_\Sigma(X)$, we have $\mathrm{sh}_\Sigma^{-1}(y) = \cap_{\varrho \in \Sigma} \mathrm{sh}_\varrho^{-1}(e_\varrho(y))$.

Let Z be a closed subvariety Z of X. If $\mathrm{sh}_\Sigma(Z)$ is a point, then $\mathrm{sh}_\varrho(Z)$ is a point for any $\varrho \in \Sigma$ by Item (a). It follows that $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite for every $\varrho \in \Sigma$.

Conversely, if $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is finite for every $\varrho \in \Sigma$, then $\mathrm{sh}_\varrho(Z)$ is a point for any $\varrho \in \Sigma$. By Item (a), $\mathrm{sh}_\Sigma(Z)$ is a point. The corollary is proved. □

3.5. On the algebraicity of the Shafarevich morphism via L^2-methods. — In Theorem 3.39, when X is compact, we proved that the image $\mathrm{Sh}_\varrho(X)$ is projective. In general, as mentioned in Remark 0.2, we propose the following conjecture.

***Conjecture 3.44* (Algebraicity of Shafarevich morphism)**
Let X, ϱ and $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ be as in Theorem 3.39. Then $\mathrm{Sh}_\varrho(X)$ is a quasi-projective normal variety and $\mathrm{sh}_\varrho : X \to \mathrm{Sh}_\varrho(X)$ is an algebraic morphism.

This conjecture seems to be a difficult problem, with the special case when ϱ arises from a $\mathbb{Z}$-VHS known as a long-standing Griffiths conjecture. In this paper, we provide confirmation of such expectations at the function field level, inspired by the work of Sommese [Som75, Som78].

We first recall the definition of (bi)meromorphic maps of complex spaces X and Y (in the sense of Remmert) with a few exceptional convenience. Let X° be an open subset of X such that $X \backslash X^\circ$ is a nowhere-dense analytic subset and suppose that a holomorphic mapping $f : X^\circ \to Y$ has been given. Then $f : X \dashrightarrow Y$ is called a meromorphic mapping if the closure Γ_f of the graph of f in $X \times Y$ is an analytic subset of $X \times Y$ and if the projection $\Gamma_f \to X$ is a proper mapping. If additionally, there is a Zariski dense open subset $X' \subset X$ such that $f|_{X'} : X' \to f(X')$ is a biholomorphism, then f is called *bimeromorphic*. It is worth noting that this definition does not require Γ_f to be proper over Y, which differs from the standard definition of bimeromorphic maps.

We present a result that is derived from [Som78, Proposition I], where the proof utilizes an elegant application of the Hörmander-Andreotti-Vesentini L^2-estimate.

Proposition 3.45. — *Let X be a smooth quasi-projective variety and let $f : X \to Y$ be a proper surjective holomorphic map onto a normal complex space Y. Let $\overline{X}$ be a smooth projective compactification of X such that $\overline{X} \backslash X$ is a simple normal crossing divisor. Assume that there exists a holomorphic line bundle L on Y equipped with a smooth hermitian metric h satisfying the following property:*

(a) f^*L extends to an algebraic line bundle $\mathcal{L}$ on $\overline{X}$.
(b) $\mathcal{L}$ has L^2-poles with respect to f^*h, i.e., for any point x in the smooth locus of D, it has an admissible coordinate $(U; z_1, \ldots, z_n)$ centered at x with $D \cap U = (z_1 = 0)$ such that $\mathcal{L}|_U$ is trivialized by a section $s \in \Gamma(U, \mathcal{L})$ and $\int_{U \backslash D} |z_1|^N |s|^2_{f^*h} i dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge i dz_n \wedge d\bar{z}_n < \infty$ for some integer $N \geq 1$.
(c) The curvature $i\Theta_h(L)$ is semipositive everywhere and strictly positive at a general smooth point of Y.

Then there exists a bimeromorphic map $h : Y \dashrightarrow W$ to a quasi-projective variety W such that $h \circ f : X \dashrightarrow W$ is rational.

As the paper by Sommese [Som78] is rather involved, for the readers' convenience, we recall briefly the ideas of the proof in [Som78].

Sketch of the proof. — After taking successive generic hyperplane sections on $\overline{X}$, we assume that there exists a proper surjective generically finite holomorphic map $g : Z \to Y$

from a complete Kähler manifold Z. By Item (c) we can choose an open set $U \subset Y^{\mathrm{reg}}$ such that

(a) $g^{-1}(U) = \cup_{i=1}^{m} U_i$ with $g|_{U_i} : U_i \to U$ is a biholomorphism;
(b) $i\Theta_h(L)$ is strictly positive at U.

We fix a point $y \in U$ and let z_i be the unique point in U_i such that $g(z_i) = y$. By applying the Hörmander L^2-estimate, we can prove that there exists integer $N_0 \geq 1$ such that for any $N \geq N_0$, the global L^2-sections $L^2(Z, K_Z \otimes g^*L^{\otimes N})$ generates 1 -jets at points $z_1, \ldots, z_m$, where $g^*L^{\otimes N}$ is equipped with the metric $g^*h^{\otimes N}$. For any $e \in L^2(Z, K_Z \otimes g^*L^{\otimes N})$, the trace map induces a section on $\widetilde{e} \in L^2(Z, \mathcal{K}_Y \otimes L^{\otimes N})$, where $\mathcal{K}_Y$ is the Grauert-Riemenschneider sheaf of Y (cf. [Som78, Lemma II-A]). Therefore, when $N \geq N_0$, the sections $L^2(Y, \mathcal{K}_Y \otimes L^{\otimes N})$ generating 1-jet at y. We then choose a finite set of sections in $L^2(Y, \mathcal{K}_Y \otimes L^{\otimes N})$ generating 1-jets at y. It thus induces a meromorphic map $h : Y \dashrightarrow \mathbb{P}^N$ such that h is immersive at a neighborhood of y.

On the other hand, by [Som78, Lemma I-C], $f^*L^2(Y, \mathcal{K}_Y \otimes L^{\otimes N})$ extends to a meromorphic section of $\Omega^k_{\overline{X}} \otimes \mathcal{L}^{\otimes N}$, where $k := \dim Y$. Therefore, by [Som75, Lemma I-E] there is a meromorphic map $p : \overline{X} \dashrightarrow \mathbb{P}^N$ such that $h \circ f = p|_X$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] & \overline{X} \arrow[d, dashed, "p"] \\ Y \arrow[r, dashed, "h"] & \mathbb{P}^N \end{tikzcd}$$

By the Chow theorem, p is rational. Let W be the image of p which is a projective variety. Then $\dim W = \dim Y$ and $h(Y) \subset W$. Since h is immersive at one point, it follows that there is a Zariski dense open set Y° such that $h|_{Y^\circ} : Y^\circ \to h(Y^\circ)$ is a biholomorphism. Therefore, $h : Y \dashrightarrow W$ is a bimeromorphic map. □

Let us apply Proposition 3.45 to study the algebraicity property of the Shafarevich morphism constructed in Theorem 3.39.

Theorem 3.46. — *Let X be a non-compact smooth quasi-projective variety and $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ be a reductive representation. Then after we replace X by some finite étale cover and ϱ by its pullback over the cover, there exists a bimeromorphic map $h : \mathrm{Sh}_\varrho(X) \dashrightarrow Y$ to a quasi-projective normal variety Y such that $h \circ \mathrm{sh}_\varrho : X \dashrightarrow Y$ is rational.*

Proof. — We replace X by a finite étale cover such that the pullback of ϱ over this étale cover is torsion free. Based on Theorem 3.39, we can then extend X to a partial projective compactification in such a way that the representation ϱ also extends, and the extended representation has infinite monodromy at infinity. Let $\mathfrak{C} \subset M_{\mathrm{B}}(X, N)(\mathbb{C})$ be the Zariski closed subset defined in (3.12). Then $\mathfrak{C}$ is a defined over $\mathbb{Q}$, and by Proposition 3.35, $\mathfrak{C}(\mathbb{C})$ is invariant under $\mathbb{C}^*$-action. Furthermore, since $[\varrho] \in \mathfrak{C}(\mathbb{C})$, $\mathfrak{C}$ has infinite monodromy at infinity. Therefore, $\mathfrak{C}$ satisfies the conditions in Theorem 3.20, and we can apply the claims in the proof of Theorem 3.20. Let $\sigma : \pi_1(X) \to \prod_{i=1}^{m} \mathrm{GL}_N(\mathbb{C})$ be the reductive representation underlying a $\mathbb{C}$-VHS constructed in Proposition 3.12 with respect to $\mathfrak{C}$. It satisfies all the properties in Items (a) to (c) in Theorem 3.20. We will use the same notations as in the proof of Theorem 3.20.

By Lemma 1.29, we can establish a family of finitely many reductive representations $\boldsymbol{\varrho} := \{\varrho_i : X \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,\ell}$ with K_i non-archimedean local fields, which satisfies the conditions $[\varrho_i] \in \mathfrak{C}(K_i)$ for every i, and $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ coincides with the reduction map $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$.

Let $\boldsymbol{\tau} := \{\varrho_i : X \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,\ell} \cup \{\sigma : \pi_1(X) \to \prod_{i=1}^{m} \mathrm{GL}_N(\mathbb{C})\}$. Let $\pi_{\boldsymbol{\tau}} : \widetilde{X}_{\boldsymbol{\tau}} \to X$ be the covering of X corresponding to the normal subgroup $\cap_{i=1,\ldots,\ell} \ker \varrho_i \cap \ker \sigma$ of $\pi_1(X)$.

Define

$$\begin{aligned}\Phi : \widetilde{X}_{\boldsymbol{\tau}} &\to S_{\boldsymbol{\varrho}} \times \mathscr{D}\\ x &\mapsto (s_{\boldsymbol{\varrho}} \circ \pi_{\boldsymbol{\tau}}(x), p(x))\end{aligned}$$

where $p : \widetilde{X}_{\boldsymbol{\tau}} \to \mathscr{D}$ is the period mapping of the $\mathbb{C}$-VHS induced by σ. Then we have

$$\begin{tikzcd}
\widetilde{X}_H \arrow[r, "\tilde{\pi}"] \arrow[d, "\mathrm{sh}_H"'] \arrow[rr, bend left, "\pi_H"] \arrow[dr, "\Psi"] & \widetilde{X}_{\boldsymbol{\tau}} \arrow[r, "\pi_{\boldsymbol{\tau}}"] \arrow[d, "\Phi"] & X \arrow[d, "s_{\boldsymbol{\varrho}}"] \\
\widetilde{S}_H \arrow[r, "g"] & S_{\boldsymbol{\varrho}} \times \mathscr{D} & S_{\boldsymbol{\varrho}}
\end{tikzcd}$$

where $\tilde{\pi}$ is a topological Galois covering.

Claim 3.47. — *Each connected component of the fiber of Φ is compact.*

Proof. — Let $(t, o) \in S_{\boldsymbol{\varrho}} \times \mathscr{D}$ be arbitrary, and consider a connected component F of $\Phi^{-1}(t, o)$. Then any connected component F' of $\tilde{\pi}^{-1}(F)$ is a connected component of $\Psi^{-1}(t, o)$, which is compact by virtual of Lemma 3.21. Therefore, $\tilde{\pi}(F') = F$ holds, implying that F is also compact. Thus, the claim follows. $\square$

As a result of Claim 3.47 and Theorem 1.30, the set $\widetilde{S}_{\boldsymbol{\tau}}$ consisting of connected components of fibers of Φ can be equipped with the structure of a complex normal space. Moreover, we have $\Phi = g_0 \circ \widetilde{s}_{\boldsymbol{\tau}}$ where $\widetilde{s}_{\boldsymbol{\tau}} : \widetilde{X}_{\boldsymbol{\tau}} \to \widetilde{S}_{\boldsymbol{\tau}}$ is a proper holomorphic fibration and $g_0 : \widetilde{S}_{\boldsymbol{\tau}} \to \widetilde{S}_{\mathfrak{C}} \times \mathscr{D}$ is a holomorphic map. By the proof of Claim 3.47, $\tilde{\pi}$ maps each fiber of $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$ to a fiber of $\widetilde{s}_{\boldsymbol{\tau}} : \widetilde{X}_{\boldsymbol{\tau}} \to \widetilde{S}_{\boldsymbol{\tau}}$. This induces a holomorphic map $\tilde{\mu} : \widetilde{S}_H \to \widetilde{S}_{\boldsymbol{\tau}}$ and we have the following commutative diagram

$$\begin{tikzcd}
\widetilde{X}_H \arrow[r, "\tilde{\pi}"] \arrow[d, "\mathrm{sh}_H"'] \arrow[rr, bend left, "\pi_H"] & \widetilde{X}_{\boldsymbol{\tau}} \arrow[r, "\pi_{\boldsymbol{\tau}}"] \arrow[d, "\widetilde{s}_{\boldsymbol{\tau}}"] & X \\
\widetilde{S}_H \arrow[r, "\tilde{\mu}"] \arrow[dr, "g"'] & \widetilde{S}_{\boldsymbol{\tau}} \arrow[d, "g_0"] & \\
& S_{\boldsymbol{\varrho}} \times \mathscr{D} &
\end{tikzcd}$$

Claim 3.48. — *$\widetilde{s}_{\boldsymbol{\tau}}$ contracts every compact subvariety of $\widetilde{X}_{\boldsymbol{\tau}}$.*

Proof. — The proof is exactly the same as Claim 3.24 and we repeat it for the sake of completeness. Let $Z \subset \widetilde{X}_{\boldsymbol{\tau}}$ be a compact irreducible subvariety. Then $W := \pi_{\boldsymbol{\tau}}(Z)$ is also a compact irreducible subvariety in X with $\dim Z = \dim W$. Hence $\mathrm{Im}\,[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(W^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(W^{\mathrm{norm}})$. Note that W can be endowed with an algebraic structure induced by X. As the natural map $Z \to W$ is finite, Z can be equipped with an algebraic structure such that the natural map $Z \to X$ is algebraic.

By the definition of $\widetilde{X}_{\boldsymbol{\tau}}$, we have $\varrho_i(\mathrm{Im}\,[\pi_1(Z) \to \pi_1(X)]) \subset \varrho_i(\mathrm{Im}\,[\pi_1(\widetilde{X}_{\boldsymbol{\tau}}) \to \pi_1(X)]) = \{1\}$ for each $\varrho_i \in \boldsymbol{\varrho}$. Hence, $\varrho_i(\mathrm{Im}\,[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X)])$ is finite which is thus bounded. By the definition of $s_{\boldsymbol{\varrho}}$, W is contained in a fiber of $s_{\boldsymbol{\varrho}}$. Consider a desingularization Z' of Z and let $i : Z' \to X$ be the natural algebraic morphism. Note that $i^*\sigma(\pi_1(Z')) = \{1\}$. It follows that the $\mathbb{C}$-VHS induced by $i^*\sigma$ is trivial. Therefore, for the period mapping $p : \widetilde{X}_{\boldsymbol{\tau}} \to \mathscr{D}$, $p(Z)$ is a point. Hence Z is contracted by $\widetilde{s}_{\boldsymbol{\tau}}$. The claim follows. $\square$

By Claim 3.48, we can apply a similar proof as in Lemma 3.25 to $\widetilde{s}_{\boldsymbol{\tau}} : \widetilde{X}_{\boldsymbol{\tau}} \to \widetilde{S}_{\boldsymbol{\tau}}$. This allows us to conclude that there is an action of $\mathrm{Aut}(\widetilde{X}_{\boldsymbol{\tau}}/X)$ on $\widetilde{S}_{\boldsymbol{\tau}}$ that is equivariant for the proper holomorphic fibration $\widetilde{s}_{\boldsymbol{\tau}} : \widetilde{X}_{\boldsymbol{\tau}} \to \widetilde{S}_{\boldsymbol{\tau}}$. This action is analytic and properly discontinuous. Taking the quotient of $\widetilde{s}_{\boldsymbol{\tau}}$ by this action, we obtain a proper holomorphic fibration $\mathrm{sh}_{\mathfrak{C}} : X \to \mathrm{Sh}_{\mathfrak{C}}(X)$ defined in the proof of Theorem 3.20, as it is also the quotient of $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$ by $\pi_1(X)/H$.

$$\begin{tikzcd}
\widetilde{X}_H \arrow[r, "\tilde{\pi}"] \arrow[d, "\mathrm{sh}_H"'] \arrow[rr, bend left, "\pi_H"] & \widetilde{X}_{\boldsymbol{\tau}} \arrow[r, "\pi_{\boldsymbol{\tau}}"] \arrow[d, "\widetilde{s}_{\boldsymbol{\tau}}"] & X \arrow[d, "\mathrm{sh}_{\mathfrak{C}}"] \arrow[dr, "\mathrm{sh}_{\varrho}"] & \\
\widetilde{S}_H \arrow[r, "\tilde{\mu}"] & \widetilde{S}_{\boldsymbol{\tau}} \arrow[r] & \mathrm{Sh}_{\mathfrak{C}}(X) \arrow[r, equal] & \mathrm{Sh}_{\varrho}(X)
\end{tikzcd}$$

It is worth noting that $\mathrm{sh}_{\mathfrak{C}} : X \to \mathrm{Sh}_{\mathfrak{C}}(X)$ coincides with the Shafarevich morphism $\mathrm{sh}_{\varrho} : X \to \mathrm{Sh}_{\varrho}(X)$, as shown in Step 1 of the proof of Proposition 3.34.

Claim 3.49. — *There exists a finite index normal subgroup N of $\mathrm{Aut}(\widetilde{X}_{\boldsymbol{\tau}}/X)$ such that its action on $\widetilde{S}_{\boldsymbol{\tau}}$ does not have any fixed point.*

Proof. — Note that $\mathrm{Aut}(\widetilde{X}_{\boldsymbol{\tau}}/X) \simeq \frac{\pi_1(X)}{\mathrm{Im}[\pi_1(\widetilde{X}_{\boldsymbol{\tau}}) \to \pi_1(X)]}$. Since $\widetilde{X}_{\boldsymbol{\tau}}$ is the covering of X corresponding to the normal subgroup $\cap_{i=1,\ldots,\ell} \ker \varrho_i \cap \ker \sigma$ of $\pi_1(X)$, it follows that $\mathrm{Aut}(\widetilde{X}_{\boldsymbol{\tau}}/X) \simeq \frac{\pi_1(X)}{\cap_{i=1,\ldots,\ell} \ker \varrho_i \cap \ker \sigma}$. Hence $\mathrm{Aut}(\widetilde{X}_{\boldsymbol{\tau}}/X)$ is finitely generated and linear. By Malcev's theorem, $\mathrm{Aut}(\widetilde{X}_{\boldsymbol{\tau}}/X)$ has a finite index normal subgroup N that is torsion free. We will prove that N acts on $\widetilde{S}_{\boldsymbol{\tau}}$ without fixed point.

Assume that there exists $\gamma \in N$ and $y \in \widetilde{S}_{\boldsymbol{\tau}}$ such that $\gamma.y = y$. Let $F := \widetilde{s}_{\boldsymbol{\tau}}^{-1}(y)$, which is a compact connected analytic subset of $\widetilde{X}_{\boldsymbol{\tau}}$ by Claim 3.47. We have $\gamma.F = F$. Since F is compact, the subgroup $\mathcal{S}$ of N that fixes F is finite. Since N is torsion-free, it follows that $\mathcal{S} = \{1\}$ and thus $\gamma = 1$. Therefore, the fixator of arbitrary point $y \in \widetilde{S}_{\boldsymbol{\tau}}$ in N can only be the identity element. Thus, the claim is proved. □

Let $Y := \widetilde{X}_{\boldsymbol{\tau}}/N$. Then $f : Y \to X$ is a finite Galois étale cover. Since $\widetilde{s}_{\boldsymbol{\tau}} : \widetilde{X}_{\boldsymbol{\tau}} \to \widetilde{S}_{\boldsymbol{\tau}}$ is $\mathrm{Aut}(\widetilde{X}_{\boldsymbol{\tau}}/X)$-equivariant, we take its quotient by N to obtain a proper holomorphic fibration $\mathrm{sh}_{f^*\varrho} : Y \to \mathrm{Sh}_{f^*\varrho}(Y)$ over a complex normal space $\mathrm{Sh}_{f^*\varrho}(Y)$. As shown in Claim 3.49, N acts on $\widetilde{S}_{\boldsymbol{\tau}}$ properly continuous and freely. Hence the covering $\widetilde{S}_{\boldsymbol{\tau}} \to \mathrm{Sh}_{f^*\varrho}(Y)$ is étale.

$$\begin{tikzcd}
\widetilde{X}_{\boldsymbol{\tau}} \arrow[r] \arrow[d, "\widetilde{s}_{\boldsymbol{\tau}}"'] \arrow[rr, bend left, "\pi_{\boldsymbol{\tau}}"] & Y \arrow[r, "f"] \arrow[d, "\mathrm{sh}_{f^*\varrho}"] & X \arrow[d, "\mathrm{sh}_{\varrho}"] \arrow[dr, "\mathrm{sh}_{\mathfrak{C}}"] & \\
\widetilde{S}_{\boldsymbol{\tau}} \arrow[r] & \mathrm{Sh}_{f^*\varrho}(Y) \arrow[r] & \mathrm{Sh}_{\varrho}(X) \arrow[r, equal] & \mathrm{Sh}_{\mathfrak{C}}(X)
\end{tikzcd}$$

Claim 3.50. — *The proper holomorphic fibration $\mathrm{sh}_{f^*\varrho} : Y \to \mathrm{Sh}_{f^*\varrho}(Y)$ is the Shafarevich morphism of $f^*\varrho$.*

Proof. — Let Z be a closed subvariety of Y. Then $W := f(Z)$ is an irreducible closed subvariety in X with $\dim Z = \dim W$. Hence $\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(W^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(W^{\mathrm{norm}})$. Since $\mathrm{Sh}_{f^*\varrho}(Y) \to \mathrm{Sh}_{\varrho}(X)$ is a finite holomorphic map, Z is contracted by $\mathrm{sh}_{f^*\varrho}$ if and only if $\mathrm{sh}_{\varrho}(W)$ is a point. This is equivalent to $\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X)])$ is finite as $\mathrm{sh}_{\varrho} : X \to \mathrm{Sh}_{\varrho}(X)$ is the Shafarevich morphism of ϱ. This, in turn, is equivalent to $f^*\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(Y)])$ is finite. The claim is proved. □

Claim 3.51. — *Each fiber of $g_0 : \widetilde{S}_{\boldsymbol{\tau}} \to S_{\boldsymbol{\varrho}} \times \mathscr{D}$ is discrete.*

Proof. — The proof is the same as in Claim 3.31. We provide it for the sake of completeness. Let $(t, o) \in S_{\boldsymbol{\varrho}} \times \mathscr{D}$ be arbitrary point and take any point $y \in g_0^{-1}((t, o))$. Then $Z := \widetilde{s}_{\boldsymbol{\tau}}^{-1}(y)$ is a connected component of the fiber $\Phi^{-1}((t, o))$, that is compact by Claim 3.47. By Theorem 1.30, Z has an open neighborhood U such that $\Phi(U)$ is a locally closed analytic subvariety of $S_{\boldsymbol{\varrho}} \times \mathscr{D}$ and $\Phi|_U : U \to \Psi(U)$ is proper. Therefore, for the Stein factorization $U \to V \xrightarrow{\pi_V} \Phi(U)$ of $\Phi|_U$, $U \to V$ coincides with $\widetilde{s}_{\boldsymbol{\tau}}|_U : U \to \widetilde{s}_{\boldsymbol{\tau}}(U)$ and $\pi_V : V \to \Phi(U)$ is finite. Note that V is an open neighborhood of y and $\pi_V : V \to \Phi(U)$ coincides with $g_0|_V : V \to S_{\boldsymbol{\varrho}} \times \mathscr{D}$. Therefore, the set $V \cap g_0^{-1}((t,o)) = V \cap (\pi_V)^{-1}(t,o)$ is finite. As a result, $g_0^{-1}((t,o))$ is discrete. The claim is proven. $\square$

For the readers' convenience, we draw a commutative diagram below.

$$\begin{tikzcd}
\widetilde{X}_{\boldsymbol{\tau}} \arrow[r] \arrow[d, "\widetilde{s}_{\boldsymbol{\tau}}"] \arrow[dd, bend right, "p"'] & Y \arrow[d, "\mathrm{sh}_{f^*\varrho}"] \arrow[dr, "s_{\boldsymbol{\varrho}}\circ f"] & \\
\widetilde{S}_{\boldsymbol{\tau}} \arrow[r] \arrow[d, "\phi"] \arrow[dr, "g_0"] & \mathrm{Sh}_{f^*\varrho}(Y) \arrow[r, "q"] & S_{\boldsymbol{\varrho}} \\
\mathscr{D} & \mathscr{D} \times S_{\mathfrak{C}} \arrow[l] &
\end{tikzcd}$$

Recall that the canonical bundle $K_{\mathscr{D}}$ of the period domain $\mathscr{D}$ is equipped with a G_0-invariant smooth metric $h_{\mathscr{D}}$, which has a positive-definite curvature in the horizontal direction. The period mapping $p : \widetilde{X}_{\boldsymbol{\tau}} \to \mathscr{D}$ of $\mathbb{C}$-VHS associated with σ induces a holomorphic map $\phi : \widetilde{S}_{\boldsymbol{\tau}} \to \mathscr{D}$ that is horizontal. Observe that ϕ is equivariant for the $\mathrm{Aut}(\widetilde{X}_{\boldsymbol{\tau}}/X)$-action. As a result, $\phi^* K_{\mathscr{D}}$ descends to a line bundle on the quotient $\mathrm{Sh}_{f^*\varrho}(Y)$, denoted by L_G. Since $\widetilde{S}_{\boldsymbol{\tau}} \to \mathrm{Sh}_{f^*\varrho}(Y)$ is étale, the smooth metric $h_{\mathscr{D}}$ induces a smooth metric h_G on L_G whose curvature form is smooth and denoted by T. Note that T is semipositive as ϕ is horizontal.

On the other hand, for the period mapping $p : \widetilde{X}_{\boldsymbol{\tau}} \to \mathscr{D}$, the pullback $p^* K_{\mathscr{D}}$ descends to a holomorphic line bundle on Y that is equal to $(\mathrm{sh}_{f^*\varrho})^* L_{\mathrm{G}}$. It is well-known that $(\mathrm{sh}_{f^*\varrho})^* L_{\mathrm{G}}$ extends to an algebraic line bundle $\mathcal{L}_1$ over $\overline{Y}$, known as the Deligne extension. According to [Den23, §2.1], $\mathcal{L}_1$ has L^2-poles with respect to the pullback metric $(\mathrm{sh}_{f^*\varrho})^* h_{\mathrm{G}}$.

Let $\overline{Y}$ be a smooth projective compactification of Y such that the boundary $D_Y := \overline{Y} \backslash Y$ is a simple normal crossing divisor. Consider the reduction map $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ of $\boldsymbol{\varrho}$. Let $\overline{S}_{\boldsymbol{\varrho}}$ be a projective compactification of $S_{\boldsymbol{\varrho}}$. Since $s_{\boldsymbol{\varrho}} \circ f : Y \to S_{\boldsymbol{\varrho}}$ is an algebraic morphism, we can blow-up D_Y such that $s_{\boldsymbol{\varrho}} \circ f$ extends to a morphism $j : \overline{Y} \to \overline{S}_{\boldsymbol{\varrho}}$. Let us choose an ample line bundle L_0 on $\overline{S}_{\boldsymbol{\varrho}}$, equipped with a smooth metric h_0 of positive-definite curvature. Let $L := q^* L_0 \otimes L_{\mathrm{G}}$, and equip it with the smooth metric $h := q^* h_0 \otimes h_{\mathrm{G}}$. It is worth noting that the algebraic line bundle $\mathcal{L} := \mathcal{L}_1 \otimes j^* L_0$ on $\overline{Y}$ extends $(\mathrm{sh}_{f^*\varrho})^* L$, and has L^2-poles with respect to $(\mathrm{sh}_{f^*\varrho})^* h$.

According to Claim 3.31, the holomorphic map $g : \widetilde{S}_H \to \mathscr{D} \times S_{\boldsymbol{\varrho}}$ has discrete fibers. Therefore, at general points on the regular locus of $\mathrm{Sh}_{f^*\varrho}(Y)$, the curvature $i\Theta_h(L)$ of (L,h) is strictly positive. Note that $i\Theta_h(L)$ is semipositive everywhere. Consequently, the conditions in Proposition 3.45 are satisfied. Thus, we can conclude that there exists a bimeromorphic map $b : \mathrm{Sh}_{f^*\varrho}(Y) \dashrightarrow Q$ to a quasi-projective variety Q such that $b \circ \mathrm{sh}_{f^*\varrho} : Y \dashrightarrow Q$ is rational. Since Y is the finite étale cover of X and $\mathrm{sh}_{f^*\varrho} : Y \to \mathrm{Sh}_{f^*\varrho}(Y)$ is the Shafarevich morphism of $f^*\varrho$ as shown in Claim 3.50, we conclude the proof of the theorem. $\square$

4. Proof of the reductive Shafarevich conjecture

The goal of this section is to provide proofs for Theorems B and C when X is a *smooth* projective variety. It is important to note that our methods differs from the approach presented in [Eys04], although we do follow the general strategy in that work.

In this section, we will use the notation $\mathcal{D}G$ to denote the derived group of any given group G. Throughout the section, our focus is on non-archimedean local fields with characteristic zero. More precisely, we consider finite extensions of $\mathbb{Q}_p$ for some prime p.

4.1. Reduction map of representation into algebraic tori. — Let X be a smooth projective variety. Let $a : X \to A$ be the Albanese morphism of X.

Lemma 4.1. — *Let $P \subset A$ be an abelian subvariety of the Albanese variety A of X and K be a non-archimedean local field. If $\tau : \pi_1(X) \to \mathrm{GL}_1(K)$ factors through $\sigma : \pi_1(A/P) \to \mathrm{GL}_1(K)$, then the Katzarkov-Eyssidieux reduction map $s_\tau : X \to S_\tau$ factors through the Stein factorization of the map $q : X \to A/P$.*

Proof. — As $\tau = q^*\sigma$, if follows that for each connected component F of the fiber of $q : X \to A/P$, $\tau(\pi_1(F)) = \{1\}$. Therefore, F is contracted by s_τ. The lemma follows. □

Lemma 4.2. — *Let $P \subset A$ be an abelian subvariety of A. Let N be a Zariski dense open set of the image $j : M_{\mathrm{B}}(A/P, 1) \to M_{\mathrm{B}}(A, 1)$ where we consider $M_{\mathrm{B}}(A/P, 1)$ and $M_{\mathrm{B}}(A, 1)$ as algebraic tori defined over $\bar{\mathbb{Q}}$. Then there are non-archimedean local fields K_i and a family of representations $\boldsymbol{\tau} := \{\tau_i : \pi_1(X) \to \mathrm{GL}_1(K_i)\}_{i=1,\ldots,m}$ such that*

- *$\tau_i \in N(K_i)$, where we use the natural identification $M^0_{\mathrm{B}}(X, 1) \simeq M_{\mathrm{B}}(A, 1)$. Here $M^0_{\mathrm{B}}(X, 1)$ denotes the connected component of $M^0_{\mathrm{B}}(X, 1)$ containing the trivial representation.*
- *The reduction map $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$ is the Stein factorization of $X \to A/P$.*
- *For the canonical current $T_{\boldsymbol{\tau}}$ defined over $S_{\boldsymbol{\tau}}$, $\{T_{\boldsymbol{\tau}}\}$ is a Kähler class.*

Proof. — Let $e_1, \ldots, e_m$ be a basis of $\pi_1(A/P) \simeq H_1(A/P, \mathbb{Z})$. Note that $\bar{\mathbb{Q}}$-scheme $M_{\mathrm{B}}(A/P, 1) \simeq (\bar{\mathbb{Q}}^\times)^m$. Denote by $S \subset U(1) \cap \bar{\mathbb{Q}}$ the set of roots of unity. Then S is Zariski dense in $\bar{\mathbb{Q}}^\times$. Since $j^{-1}(N)$ is a Zariski dense open set of $M_{\mathrm{B}}(A/P, 1)$, it follows that there are $\{a_{ij}\}_{i,j=1,\ldots,m} \in \bar{\mathbb{Q}}^\times$ and representations $\{\varrho_i : \pi_1(A/P) \to \bar{\mathbb{Q}}^\times\}_{i=1,\ldots,m}$ defined by $\varrho_i(e_j) = a_{ij}$ such that

- $[\varrho_i] \in j^{-1}(N)(\bar{\mathbb{Q}})$;
- If $i = j$, $a_{ij} \in \bar{\mathbb{Q}}^\times \setminus U(1)$;
- If $i \neq j$, $a_{ij} \in S$.

Consider a number field k_i containing $a_{i1}, \ldots, a_{im}$ endowed with a discrete non-archimedean valuation $v_i : k_i \to \mathbb{R}$ such that $v_i(a_{ii}) \neq 0$. Then $v_i(a_{ij}) = 0$ for every $j \neq i$. Indeed, for every $j \neq i$, since a_{ij} is a root of unity, there exists $\ell \in \mathbb{Z}_{>0}$ such that $a_{ij}^\ell = 1$. It follows that $0 = v(a_{ij}^\ell) = \ell v(a_{ij})$. Let K_i be the non-archimedean local field which is the completion of k_i with respect to v_i. It follows that each $\varrho_i : \pi_1(A/P) \to K_i^\times$ is unbounded. Consider $\nu_i : \pi_1(A/P) \to \mathbb{R}$ by composing ϱ_i with $v_i : K_i^\times \to \mathbb{R}$. Then $\{\nu_1, \ldots, \nu_m\} \subset H^1(A/P, \mathbb{R})$ is a basis for the $\mathbb{R}$-linear space $H^1(A/P, \mathbb{R})$. It follows that $\nu_i(e_j) = \delta_{ij}$ for any i, j. Let $\eta_i \in H^0(A/P, \Omega^1_{A/P})$ be the $(1,0)$-part of the Hodge decomposition of ν_i. Therefore, $\{\eta_1, \ldots, \eta_m\}$ spans the $\mathbb{C}$-linear space $H^0(A/P, \Omega^1_{A/P})$. Hence $\sum_{i=1}^m i\eta_i \wedge \overline{\eta_i}$ is a Kähler form on A/P. Let $\tau_i : \pi_1(X) \to K_i^\times$ be the composition of ϱ_i with $\pi_1(X) \to \pi_1(A/P)$.

Let $q : A \to A/P$ be the quotient map. Let P' the largest abelian subvariety of A such that $q^*\eta_i|_{P'} \equiv 0$ for each i. Since $\{\eta_1, \ldots, \eta_m\}$ spans $H^0(B, \Omega^1_B)$, it follows that $P' = P$. Therefore, the reduction map $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$ is the Stein factorization of $X \to A/P$ with $g : S_{\boldsymbol{\tau}} \to A/P$ be the finite morphism. According to Definition 1.24, $T_{\boldsymbol{\tau}} = g^* \sum_{i=1}^m i\eta_i \wedge \overline{\eta_i}$. Since $\sum_{i=1}^m i\eta_i \wedge \overline{\eta_i}$ is a Kähler form on A/P, it follows that $\{T_{\boldsymbol{\tau}}\}$ is a Kähler class by Theorem 1.13. The lemma is proved. □

Corollary 4.3. — *Let X be a smooth projective variety. If $\mathfrak{C} \subset M_{\mathrm{B}}(X, 1)$ is an absolutely constructible subset. Consider the reduction map $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ defined in Definition 3.1. Then there is a family of representations $\boldsymbol{\varrho} := \{\varrho_i : \pi_1(X) \to \mathrm{GL}_1(K_i)\}_{i=1,\ldots,\ell}$ where K_i are non-archimedean local fields such that*

- *For each $i = 1, \ldots, \ell$, $\varrho_i \in \mathfrak{C}(K_i)$;*
- *The reduction map $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ of $\boldsymbol{\varrho}$ coincides with $s_{\mathfrak{C}}$.*
- *For the canonical current $T_{\boldsymbol{\varrho}}$ defined over $S_{\mathfrak{C}}$, $\{T_{\boldsymbol{\varrho}}\}$ is a Kähler class.*

Proof. — Let A be the Albanese variety of X. Since $\mathfrak{C} \subset M_{\mathrm{B}}(X,1)$ is an absolute constructible subset, by Theorem 1.19, there are abelian subvarieties $P_i \subset A$ and torsion points $v_i \in M_{\mathrm{B}}(X,1)(\bar{\mathbb{Q}})$ such that $\mathfrak{C} = \cup_{i=1}^m v_i.N_i^\circ$; where N_i is the image in $M_{\mathrm{B}}^0(X,1) \simeq M_{\mathrm{B}}(A,1)$ of the natural morphism $M_{\mathrm{B}}(A/P_i,1) \to M_{\mathrm{B}}(A,1)$ and N_i° is a Zariski dense open subset of N_i. Let k be a number field such that $v_i \in M_{\mathrm{B}}(X,1)(k)$ for each i.

Claim 4.4. — *Denote by $P := \cap_{i=1}^m P_i$. Then $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ is the Stein factorization of $X \to A/P$.*

Proof. — Let $\tau : \pi_1(X) \to \mathrm{GL}_1(K)$ be a reductive representation with K a non-archimedean local field such that $\tau \in \mathfrak{C}(K)$. Note that the reduction map s_τ is the same if we replace K by a finite extension. We thus can assume that $k \subset K$. Note that there exists some $i \in \{1,\ldots,\ell\}$ such that $[v_i^{-1}.\tau] \in N_i(K)$. Write $\varrho := v_i^{-1}.\tau$. Since v_i is a torsion element, it follows that $v_i(\pi_1(X))$ is finite, and thus the reduction map s_ϱ coincides with s_τ. Since ϱ factors through $\pi_1(A/P_i) \to \mathrm{GL}_1(K)$, by Lemma 4.1 s_ϱ factors through the Stein factorization of $X \to A/P_i$. Hence s_ϱ factors through the Stein factorization of $X \to A/P$. By Definition 3.1, it follows that $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ factors through the Stein factorization of $X \to A/P$.

Fix any i. By Lemma 4.2 there are non-archimedean local fields K_j and a family of reductive representations $\boldsymbol{\tau} := \{\tau_j : \pi_1(X) \to \mathrm{GL}_1(K_j)\}_{j=1,\ldots,n}$ such that

— $\tau_j \in N_i^\circ(K_j)$.
— The reduction map $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$ is the Stein factorization of $X \to A/P_i$.
— For the canonical current $T_{\boldsymbol{\tau}}$ over $S_{\boldsymbol{\tau}}$, $\{T_{\boldsymbol{\tau}}\}$ is a Kähler class.

We can replace K_i by a finite extension such that $k \subset K_i$ for each K_i. Then $v_i.\tau_i \in \mathfrak{C}(K_i)$ for every i. Note that the Katzarkov-Eyssidieux reduction map $s_{v_i.\tau_j} : X \to S_{v_i.\tau_j}$ coincides with $s_{\tau_j} : X \to S_{\tau_j}$. Therefore, the Stein factorization of $X \to A/P_i$ factors through $s_{\mathfrak{C}}$. Since this holds for each i, it follows that the Stein factorization $X \to A/P_1 \times \cdots \times A/P_m$ factors through $s_{\mathfrak{C}}$. Note that the Stein factorization $X \to A/P_1 \times \cdots \times A/P_m$ coincides with the Stein factorization of $X \to A/P$. Therefore, the Stein factorization of $X \to A/P$ factors through $s_{\mathfrak{C}}$. The claim is proved. □

By the above arguments, for each i, there exists a family of reductive representations into non-archimedean local fields $\boldsymbol{\varrho}_i := \{\varrho_{ij} : \pi_1(X) \to \mathrm{GL}_1(K_{ij})\}_{j=1,\ldots,k_i}$ such that

— $\varrho_{ij} \in \mathfrak{C}(K_{ij})$
— $s_{\boldsymbol{\varrho}_i} : X \to S_{\boldsymbol{\varrho}_i}$ is the Stein factorization of $X \to A/P_i$
— For the canonical current $T_{\boldsymbol{\varrho}_i}$ defined over $S_{\boldsymbol{\varrho}_i}$, $\{T_{\boldsymbol{\varrho}_i}\}$ is a Kähler class.

By the above claim, we know that $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ is the Stein factorization of $X \to S_{\boldsymbol{\varrho}_1} \times \cdots \times S_{\boldsymbol{\varrho}_m}$. Then for the representation $\boldsymbol{\varrho} := \{\varrho_{ij} : \pi_1(X) \to \mathrm{GL}_1(K_{ij})\}_{i=1,\ldots,m; j=1,\ldots,k_i}$, $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ is the Stein factorization of $X \to A/P$ hence $s_{\boldsymbol{\varrho}}$ coincides with $s_{\mathfrak{C}}$. Moreover, the canonical current $T_{\boldsymbol{\varrho}} = \sum_{i=1}^m g_i^* T_{\boldsymbol{\varrho}_i}$ where $g_i : S_{\mathfrak{C}} \to S_{\boldsymbol{\varrho}_i}$ is the natural map. As $S_{\mathfrak{C}} \to S_{\boldsymbol{\varrho}_1} \times \cdots \times S_{\boldsymbol{\varrho}_m}$ is finite, by Theorem 1.13 $\{T_{\boldsymbol{\varrho}}\}$ is Kähler. □

Let us prove the main result in this subsection.

Theorem 4.5. — *Let X be a smooth projective variety and let T be an algebraic tori defined over some number field k. Let $\mathfrak{C} \subset M_{\mathrm{B}}(X,T)(\mathbb{C})$ be an absolutely constructible subset. Consider the reduction map $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$. Then there is a family of reductive representations $\boldsymbol{\tau} := \{\tau_i : \pi_1(X) \to T(K_i)\}_{i=1,\ldots,N}$ where K_i are non-archimedean local fields containing k such that*

— *For each $i = 1,\ldots,N$, $[\tau_i] \in \mathfrak{C}(K_i)$;*
— *The reduction map $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$ of $\boldsymbol{\tau}$ coincides with $s_{\mathfrak{C}}$.*
— *For the canonical current $T_{\boldsymbol{\tau}}$ over $S_{\mathfrak{C}}$ defined in Definition 1.24, $\{T_{\boldsymbol{\tau}}\}$ is a Kähler class.*

Proof. — We replace k by a finite extension such that T is split over k. Then we have $T \simeq \mathbb{G}_{m,k}^\ell$. Note that this does not change the reduction map $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$. We take

$p_i : T \to \mathbb{G}_{m,k}$ to be the i-th projection which is a k-morphism. It induces a morphism of k-schemes $\psi_i : M_{\mathrm{B}}(X, T) \to M_{\mathrm{B}}(X, \mathrm{GL}_1)$. By Theorem 1.20, $\mathfrak{C}_i := \psi_i(\mathfrak{C})$ is also an absolutely constructible subset. Consider the reduction maps $\{s_{\mathfrak{C}_i} : X \to S_{\mathfrak{C}_i}\}_{i=1,\ldots,\ell}$ defined by Definition 3.1.

Claim 4.6. — *$s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ is the Stein factorization of $s_{\mathfrak{C}_1} \times \cdots \times s_{\mathfrak{C}_\ell} : X \to S_{\mathfrak{C}_1} \times \cdots \times S_{\mathfrak{C}_\ell}$.*

Proof. — Let $\varrho : \pi_1(X) \to T(K)$ be any reductive representation where K is a non-archimedean local field containing k such that $[\varrho] \in \mathfrak{C}(K)$. Write $\varrho_i = p_i \circ \varrho : \pi_1(X) \to \mathrm{GL}_1(K)$. Then $[\varrho_i] = \psi_i([\varrho]) \in \mathfrak{C}_i(K)$. Note that for any subgroup $\Gamma \subset \pi_1(X)$, $\varrho(\Gamma)$ is bounded if and only if $\varrho_i(\Gamma)$ is bounded for any i. Therefore, $s_\varrho : X \to S_\varrho$ is the Stein factorization of $X \to S_{\varrho_1} \times \cdots \times S_{\varrho_\ell}$. Hence $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ factors through the Stein factorization of $X \to S_{\mathfrak{C}_1} \times \cdots \times S_{\mathfrak{C}_\ell}$.

On the other hand, consider any $\varrho_i \in \mathfrak{C}_i(K)$ where K is a non-archimedean local field containing k. Then there is a finite extension L of K such that

— there is a reductive representation $\varrho : \pi_1(X) \to T(L)$ with $[\varrho] \in \mathfrak{C}(L)$;
— $p_i \circ \varrho = \varrho_i$.

By the above argument, $s_{\varrho_i} : X \to S_{\varrho_i}$ factors through $s_\varrho : X \to S_\varrho$. Note that s_ϱ factors through $s_{\mathfrak{C}}$. It follows that the Stein factorization of $X \to S_{\mathfrak{C}_1} \times \cdots \times S_{\mathfrak{C}_\ell}$ factors through $s_{\mathfrak{C}}$. The claim is proved. □

We now apply Corollary 4.3 to conclude that for each i, there exists a family of reductive representations into non-archimedean local fields $\boldsymbol{\varrho}_i := \{\varrho_{ij} : \pi_1(X) \to \mathrm{GL}_1(K_{ij})\}_{j=1,\ldots,k_i}$ such that

— $\varrho_{ij} \in \mathfrak{C}_i(K_{ij})$;
— The reduction map $s_{\boldsymbol{\varrho}_i} : X \to S_{\boldsymbol{\varrho}_i}$ of $\boldsymbol{\varrho}_i$ coincides with $s_{\mathfrak{C}_i} : X \to S_{\mathfrak{C}_i}$;
— for the canonical current $T_{\boldsymbol{\varrho}_i}$ defined over $S_{\boldsymbol{\varrho}_i}$, $\{T_{\boldsymbol{\varrho}_i}\}$ is a Kähler class.

Denote by $\boldsymbol{\varrho} := \{\varrho_{ij}\}_{i=1,\ldots,\ell;j=1,\ldots,k_i}$. Then $s_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ coincides with $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ by the above claim. Then $T_{\boldsymbol{\varrho}}$ is a Kähler class.

By the definition of $\mathfrak{C}_i$, we can find a finite extension L_{ij} of K_{ij} such that

— there is a reductive representation $\tau_{ij} : \pi_1(X) \to T(L_{ij})$ with $[\tau_{ij}] \in \mathfrak{C}(L_{ij})$;
— $p_i \circ \tau_{ij} = \varrho_{ij}$.

Therefore, for the family $\boldsymbol{\tau} := \{\tau_{ij}\}_{i=1,\ldots,\ell;j=1,\ldots,k_i}$, $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$ coincides with $s_{\mathfrak{C}}$ by the above claim. Note that for any i, j, there exists an morphism $e_{ij} : S_{\tau_{ij}} \to S_{\varrho_{ij}}$ such that $s_{\varrho_{ij}} : X \to S_{\varrho_{ij}}$ factors through e_{ij}. We also note that $e_{ij}^* T_{\varrho_{ij}} \leq T_{\tau_{ij}}$ for the canonical currents. It follows that $T_{\boldsymbol{\varrho}} \leq T_{\boldsymbol{\tau}}$ (note that $S_{\boldsymbol{\tau}} = S_{\boldsymbol{\varrho}} = S_{\mathfrak{C}}$). Therefore, $\{T_{\boldsymbol{\tau}}\}$ is a Kähler class. We prove the theorem. □

4.2. Some criterion for representation into tori.

— We recall a lemma in [CDY22, Lemma 5.3].

Lemma 4.7. — *Let G be an almost simple algebraic group over the non-archimedean local field K. Let $\Gamma \subset G(K)$ be a finitely generated subgroup so that*

— *it is a Zariski dense subgroup in G,*
— *it is not contained in any bounded subgroup of $G(K)$.*

Let Υ be a normal subgroup of Γ which is bounded. *Then Υ must be finite.*

This lemma enables us to prove the following result.

Lemma 4.8. — *Let G be a reductive algebraic group over the non-archimedean local field K of characteristic zero. Let X be a projective manifold and let $\varrho : \pi_1(X) \to G(K)$ be a Zariski dense representation. If $\varrho(\mathcal{D}\pi_1(X))$ is bounded, then after replacing K by some finite extension, for the reductive representation $\tau : \pi_1(X) \to G/\mathcal{D}G(K)$ which is* *the composition of ϱ with $G \to G/\mathcal{D}G$, the reduction map $s_\tau : X \to S_\tau$ coincides with $s_\varrho : X \to S_\varrho$.*

Proof. — Since G is reductive, then after replacing K by a finite extension, there is an isogeny $G \to H_1 \times \cdots \times H_k \times T$, where H_i are almost simple algebraic groups over K and $T = G/\mathcal{D}G$ is an algebraic tori over K.Write $G' := H_1 \times \cdots \times H_k \times T$. We denote by $\varrho' : \pi_1(X) \to G'(K)$ the induced representation by the above isogeny.

Claim 4.9. — *The Katzarkov-Eyssidieux reduction map $s_\varrho : X \to S_\varrho$ coincides with $s_{\varrho'} : X \to S_{\varrho'}$.*

Proof. — It suffices to prove that, for any subgroup Γ of $\pi_1(X)$, $\varrho(\Gamma)$ is bounded if and only if $\varrho'(\Gamma)$ is bounded. Note that we have the following short exact sequence of algebraic groups

$$0 \to \mu \to G \to G' \to 0$$

where μ is finite. Then we have

$$0 \to \mu(K) \to G(K) \xrightarrow{f} G'(K) \to H^1(K, \mu),$$

where $H^1(K, \mu)$ is the Galois cohomology. Note that $\mu(K)$ is finite. Since K is a finite extension of some $\mathbb{Q}_p$, it follows that $H^1(K, \mu)$ is also finite. Therefore, $f : G(K) \to G'(K)$ has finite kernel and cokernel. Therefore, $\varrho(\Gamma)$ is bounded if and only if $\varrho'(\Gamma)$ is bounded. □

Set $\Gamma := \varrho'(\pi_1(X))$ and $\Upsilon := \varrho'(\mathcal{D}\pi_1(X))$. Let $\Upsilon_i \subset H_i(K)$ and Γ_i be the image of Υ and Γ under the projection $G(K) \to H_i(K)$. Then Γ_i is Zariski dense in H_i and $\Upsilon_i \triangleleft \Gamma_i$ is also bounded. Furthermore, $\mathcal{D}\Gamma_i = \Upsilon_i$.

Claim 4.10. — *Γ_i is bounded for every i.*

Proof. — Assuming a contradiction, let's suppose that some Γ_i is unbounded. Since $\Upsilon_i \triangleleft \Gamma_i$ and Υ_i is bounded, we can refer to Lemma 4.7 which states that Υ_i must be finite. We may replace X with a finite étale cover, allowing us to assume that Υ_i is trivial. Consequently, Γ_i becomes abelian, which contradicts the fact that Γ_i is Zariski dense in the almost simple algebraic group H_i. □

Based on the previous claim, it follows that the induced representations $\tau_i : \pi_1(X) \to H_i(K)$ are all bounded for every i. Consequently, they do not contribute to the reduction map of $s_{\varrho'} : X \to S_{\varrho'}$. Therefore, the only contribution to $s_{\varrho'}$ comes from $\tau : \pi_1(X) \to T(K)$, where τ is the composition of $\varrho : \pi_1(X) \to G(K)$ and $G(K) \to T(K)$.

According to Claim 4.9, we can conclude that s_ϱ coincides with the reduction map $s_\tau : X \to S_\tau$ of $\tau : \pi_1(X) \to T(K)$. This establishes the lemma. □

4.3. Eyssidieux-Simpson Lefschetz theorem and its application. — Let X be a compact Kähler manifold and let $V \subset H^0(X, \Omega_X^1)$ be a $\mathbb{C}$-subspace. Let $a : X \to \mathcal{A}_X$ be the Albanese morphism of X. Note that $a^* : H^0(\mathcal{A}_X, \Omega_{\mathcal{A}_X}^1) \to H^0(X, \Omega_X^1)$ is an isomorphism. Write $V' := (a^*)^{-1}(V)$. Define $B(V) \subset \mathcal{A}_X$ to be the largest abelian subvariety of $\mathcal{A}_X$ such that $\eta|_{B(V)} = 0$ for every $\eta \in V'$. Set $\mathcal{A}_{X,V} := \mathcal{A}_X/B(V)$. The *partial Albanese morphism associated with* V is the composition of a with the quotient map $\mathcal{A}_X \to \mathcal{A}_{X,V}$, denoted by $g_V : X \to \mathcal{A}_{X,V}$. Note that there exists $V_0 \subset H^0(\mathcal{A}_{X,V}, \Omega^1_{\mathcal{A}_{X,V}})$ with $\dim_{\mathbb{C}} V_0 = \dim_{\mathbb{C}} V$ such that $g_V^* V_0 = V$. Let $\widetilde{\mathcal{A}_{X,V}} \to \mathcal{A}_{X,V}$ be the universal covering and let X_V be $X \times_{\mathcal{A}_{X,V}} \widetilde{\mathcal{A}_{X,V}}$. Note that V_0 induces a natural linear map $\widetilde{\mathcal{A}_{X,V}} \to V_0^*$. Its composition with $X_V \to \widetilde{\mathcal{A}_{X,V}}$ and $g_V^* : V_0 \to V$ gives rise to a holomorphic map

$$\widetilde{g}_V : X_V \to V^*. \tag{4.1}$$

Let $f : X \to S$ be the Stein factorization of $g_V : X \to \mathcal{A}_{X,V}$ with $q : S \to \mathcal{A}_{X,V}$ the finite morphism. Set $\mathbb{V} := q^* V_0$.

Definition 4.11. — V is called *perfect* if for any closed subvariety $Z \subset S$ of dimension $d \geq 1$, one has $\mathrm{Im}[\Lambda^d \mathbb{V} \to H^0(Z, \Omega_Z^d)] \neq 0$.

The terminology of "perfect V" in Definition 4.11 is called "SSKB factorisable" in [Eys04, Lemme 5.1.6].

Let us recall the following Lefschetz theorem by Eyssidieux, which is a generalization of previous work by Simpson [Sim92]. This theorem plays a crucial role in the proofs of Theorems B and C.

Theorem 4.12 **([Eys04, Lemme 5.1.22]).** — *Let X be a compact Kähler normal space and let $V \subset H^0(X, \Omega_X^1)$ be a subspace. Assume that*

$$\mathrm{Im}\,[\Lambda^{\dim V} V \to H^0(X, \Omega_X^{\dim V})] = \eta \neq 0.$$

Set $(\eta = 0) = \cup_{i=1}^k Z_k$ where Z_i are proper closed subvarieties of X. For each Z_i, denote by $V_i := \mathrm{Im}[V \to H^0(Z_i, \Omega_{Z_i})]$. Assume that V_i is perfect for each i. Then there are two possibilities which exclude each other:

— *either V is perfect;*
— *or for the holomorphic map $\widetilde{g}_V : X_V \to V^*$ defined as (4.1), $(X_V, \widetilde{g}_V^{-1}(t))$ is 1-connected for any $t \in V^*$; i.e. $\widetilde{g}_V^{-1}(t)$ is connected and $\pi_1(\widetilde{g}_V^{-1}(t)) \to \pi_1(X_V)$ is surjective.*

We need the following version of the Castelnuovo-De Franchis theorem.

Theorem 4.13 **(Castelnuovo-De Franchis).** — *Let X be a compact Kähler normal space and let $W \subset H^0(X, \Omega_X)$ be the subspace of dimension $d \geq 2$ such that*

— $\mathrm{Im}\big(\Lambda^d W \to H^0(X, \Omega_X^d)\big) = 0$*;*
— *for every hyperplane $W' \subset W$, $\mathrm{Im}\big(\Lambda^{d-1} W' \to H^0(X, \Omega_X^{d-1})\big) \neq 0$.*

Then there is a projective normal variety S of dimension $d-1$ and a fibration $f : X \to S$ such that $W \subset f^* H^0(S, \Omega_S)$.

To apply Theorem 4.13, we need to show the existence of a linear subspace $W \subset H^0(X, \Omega_X)$ as in the theorem.

Lemma 4.14. — *Let X be a projective normal variety and let $V \subset H^0(X, \Omega_X)$. Let r be the largest integer such that $\mathrm{Im}\,[\Lambda^r V \to H^0(X, \Omega_X^r)] \neq 0$. Assume that $r < \dim_{\mathbb{C}} V$. There exists $W \subset H^0(X, \Omega_X)$ such that*

(i) $2 \leq \dim W \leq r + 1$.
(ii) $\mathrm{Im}\,[\Lambda^{\dim W} W \to H^0(X, \Omega_X^{\dim W})] = 0$*;*
(iii) *for every hyperplane $W' \subsetneq W$, we always have* $\mathrm{Im}\,[\Lambda^{\dim W - 1} W' \to H^0(X, \Omega_X^{\dim W - 1})] \neq 0$.

Proof. — By our assumption there exist $\{\omega_1, \ldots, \omega_r\} \subset V$ such that $\omega_1 \wedge \cdots \wedge \omega_r \neq 0$. Let $W_0 \subset V$ be the subspace generated by $\{\omega_1, \ldots, \omega_r\}$. Since $r < \dim_{\mathbb{C}} V$, there exists $\omega \in V \setminus W_0$.

Pick a point $x \in X$ such that $\omega_1 \wedge \cdots \wedge \omega_r(x) \neq 0$. Then there exists a coordinate system $(U; z_1, \ldots, z_n)$ centered at x such that $dz_i = \omega_i$ for $i = 1, \ldots, r$. Write $\omega = \sum_{i=1}^n a_i(z) dz_i$. By our choice of r, we have $\omega_1 \wedge \cdots \wedge \omega_r \wedge \omega = 0$. It follows that

— $a_j(z) = 0$ for $j = r+1, \ldots, n$;
— at least one of $a_1(z), \ldots, a_r(z)$ is not constant.

Let $k+1$ be the transcendental degree of $\{1, a_1(z), \ldots, a_r(z)\} \subset \mathbb{C}(U)$. Then $k \geq 1$. We assume that $1, a_1(z), \ldots, a_k(z)$ is linearly independent for the transcendental extension $\mathbb{C}(U)/\mathbb{C}$. One can check by an easy linear algebra that the subspace W generated $\{\omega_1, \ldots, \omega_k, \omega\}$ is an element of E. The lemma is proved. □

Lemma 4.15. — *Let X be a projective normal variety and let $V \subset H^0(X, \Omega_X)$. Let r be the largest integer such that $\operatorname{Im}[\Lambda^r V \to H^0(X, \Omega_X^r)] \neq 0$, which will be called* generic rank *of V. Consider the partial Albanese morphism $g_V : X \to \mathcal{A}_{X,V}$ induced by V. Let $V_0 \subset H^0(\mathcal{A}_{X,V}, \Omega^1_{\mathcal{A}_{X,V}})$ be the linear subspace such that $g_V^* V_0 = V$. Let $f : X \to S$ be the Stein factorization of g_V with $q : S \to \mathcal{A}_{X,V}$ the finite morphism. Consider $\mathbb{V} := q^* V_0$. Assume that*

$$\operatorname{Im}[\Lambda^{\dim Z}\mathbb{V} \to H^0(Z, \Omega_Z^{\dim Z})] \neq 0$$

for every proper closed subvariety $Z \subsetneq S$. Then there are two possibilities.

— *either*

$$\operatorname{Im}[\Lambda^{\dim S}\mathbb{V} \to H^0(S, \Omega_S^{\dim S})] \neq 0;$$

— *or $r = \dim_{\mathbb{C}} V$.*

Proof. — Assume that both

$$\operatorname{Im}[\Lambda^{\dim S}\mathbb{V} \to H^0(S, \Omega_S^{\dim S})] = 0,$$

and $r < \dim_{\mathbb{C}} V$. Therefore, $r < \dim S \leq \dim X$. By Lemma 4.14 there is a subspace $W \subset V$ with $\dim_{\mathbb{C}} W = k + 1 \leq r + 1$ such that $\operatorname{Im}[\Lambda^{\dim W} W \to H^0(X, \Omega_Y^{\dim W})] = 0$, and for any subspace $W' \subsetneq W$, we always have $\operatorname{Im}[\Lambda^{\dim W'} W' \to H^0(X, \Omega_X^{\dim W'})] \neq 0$. By our assumption, we have $\dim_{\mathbb{C}} W \leq \dim X$. By Theorem 4.13, there is a fibration $p : X \to B$ with B a projective normal variety with $\dim B = \dim W - 1 \leq \dim X - 1$ such that $W \subset p^* H^0(B, \Omega_B^1)$. In particular, the generic rank of the forms in W is $\dim W - 1$. Consider the partial Albanese morphism $g_W : X \to \mathcal{A}_{X,W}$ associated with W. We shall prove that p can be made as the Stein factorisation of g_W.

Note that each fiber of p is contracted by g_W. Therefore, we have a factorisation $X \xrightarrow{p} B \xrightarrow{h} \mathcal{A}_{X,W}$. Note that there exists a linear space $W_0 \subset H^0(\mathcal{A}_{X,W}, \Omega^1_{\mathcal{A}_{X,W}})$ such that $W = g_W^* W_0$. If $\dim h(B) < \dim B$, then the generic rank of W is less or equal to $\dim h(B)$. This contradicts with Theorem 4.13. Therefore, $\dim h(B) = \dim B$. Let $X \xrightarrow{p'} B' \to \mathcal{A}_{X,W}$ be the Stein factorisation of g_W. Then there exists a birational morphism $\nu : B \to B'$ such that $p' = \nu \circ p$. We can thus replace B by B', and p by p'.

Recall that $f : X \to S$ is the Stein factorisation of the partial Albanese morphism $g_V : X \to \mathcal{A}_{X,V}$ associated with V. As g_W factors through the natural quotient map $\mathcal{A}_{X,V} \to \mathcal{A}_{X,W}$, it follows that $p : X \to B$ factors through $X \xrightarrow{f} S \xrightarrow{\nu} B$.

Assume that $\dim S = \dim B$. Then ν is birational. Since $\dim B = \dim W - 1$ and the generic rank of W is $\dim W - 1$, it follows that

$$\operatorname{Im}[\Lambda^{\dim S}\mathbb{V} \to H^0(S, \Omega_S^{\dim S})] \neq 0.$$

This contradicts with our assumption at the beginning. Hence $\dim S > \dim B$.

Let Z be a general fiber of ν which is positive-dimensional. Since $W \subset p^* H^0(B, \Omega_B^1)$, and we have assumed that the generic rank of $\mathbb{V}$ is less than $\dim S$, it follows that the generic rank of $\operatorname{Im}[\mathbb{V} \to H^0(Z, \Omega_Z^1)]$ is less than $\dim Z$. This implies that

$$\operatorname{Im}[\Lambda^{\dim Z}\mathbb{V} \to H^0(Z, \Omega_Z^{\dim Z})] = 0,$$

which contradicts with our assumption. Therefore we obtain a contradiction. The lemma is proved. □

Remark 4.16. — Let Y be a normal projective variety. Let $\boldsymbol{\varrho} = \{\varrho_i : \pi_1(Y) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,k}$ be a family of reductive representations where K_i are non-archimedean local field. Let $\pi : X \to Y$ be a Galois cover dominating all spectral covers induced by ϱ_i. Let $V \subset H^0(X, \Omega_X)$ be the set of all spectral forms (cf. § 1.9 for definitions). We use the same notations as in Lemma 4.15. Considering Katzarkov-Eyssidieux reduction maps $s_{\boldsymbol{\varrho}} : Y \to S_{\boldsymbol{\varrho}}$ and $s_{\pi^*\boldsymbol{\varrho}} : X \to S_{\pi^*\boldsymbol{\varrho}}$. One can check that, for every closed subvariety $Z \subset S_{\boldsymbol{\varrho}}$, $\{T_{\boldsymbol{\varrho}}^{\dim Z}\} \cdot Z > 0$ if and only if for any closed subvariety $W \subset S_{\pi^*\boldsymbol{\varrho}}$ dominating Z under $\sigma_\pi : S_{\pi^*\boldsymbol{\varrho}} \to S_{\boldsymbol{\varrho}}$ defined in (1.2), one has

$$\operatorname{Im}[\Lambda^{\dim W}\mathbb{V} \to H^0(W, \Omega_W^{\dim W})] \neq 0.$$

In particular, V is perfect if and only if $\{T_{\boldsymbol{\varrho}}\}$ is a Kähler class by Theorem 1.13.

Theorem 4.17. — *Let X be a smooth projective variety and let $\boldsymbol{\varrho} := \{\varrho_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,k}$ be a family of reductive representations where K_i is a non-archimedean local field. Let $S_{\boldsymbol{\varrho}} : X \to S_{\boldsymbol{\varrho}}$ be the Katzarkov-Eyssidieux reduction map. Let $T_{\boldsymbol{\varrho}}$ be the canonical $(1,1)$-current on $S_{\boldsymbol{\varrho}}$ associated with $\boldsymbol{\varrho}$ defined in Definition 1.24. Denote by H_i the Zariski closure of $\varrho_i(\pi_1(X))$. Assume that for any proper closed subvariety $\Sigma \subsetneq S$, one has $\{T_{\boldsymbol{\varrho}}\}^{\dim \Sigma} \cdot \Sigma > 0$. Then*

— *either $\{T_{\boldsymbol{\varrho}}\}^{\dim S_{\boldsymbol{\varrho}}} \cdot S_{\boldsymbol{\varrho}} > 0$;*
— *or the reduction map $s_{\sigma_i} : X \to S_{\sigma_i}$ coincides with $s_{\varrho_i} : X \to S_{\varrho_i}$ for each i, where $\sigma_i : \pi_1(X) \to (H_i/\mathcal{D}H_i)(K_i)$ is the composition of ϱ_i with the group homomorphism $H_i \to H_i/\mathcal{D}H_i$.*

Proof. — Assume that $\{T_{\boldsymbol{\varrho}}\}^{\dim S_{\boldsymbol{\varrho}}} \cdot S_{\boldsymbol{\varrho}} = 0$. Let $Y \to X$ be a Galois cover which dominates all spectral covers of ϱ_i. We pull back all the spectral one forms on Y to obtain a subspace $V \subset H^0(Y, \Omega_Y^1)$. Consider the partial Albanese morphism $g_V : Y \to \mathcal{A}_{Y,V}$ associated to V, then $s_{\pi^*\boldsymbol{\varrho}} : Y \to S_{\pi^*\boldsymbol{\varrho}}$ is its Stein factorization with $q : S_{\pi^*\boldsymbol{\varrho}} \to \mathcal{A}_{Y,V}$ the finite morphism. Note that there is a $\mathbb{C}$-linear subspace $\mathbb{V} \subset H^0(S_{\pi^*\boldsymbol{\varrho}}, \Omega^1_{S_{\pi^*\boldsymbol{\varrho}}})$ such that $s^*_{\pi^*\boldsymbol{\varrho}}\mathbb{V} = V$.

$$\begin{tikzcd}
Y \arrow[r, "\pi"] \arrow[d, "s_{\pi^*\boldsymbol{\varrho}}"] \arrow[dd, bend right=60, "s_{\pi^*\varrho_i}"'] & X \arrow[d, "s_{\boldsymbol{\varrho}}"] \arrow[dd, bend left=60, "s_{\varrho_i}"] \\
S_{\pi^*\boldsymbol{\varrho}} \arrow[r, "\sigma_\pi"] \arrow[d, "q_i"] & S_{\boldsymbol{\varrho}} \arrow[d, "p_i"'] \\
S_{\pi^*\varrho_i} \arrow[r] & S_{\varrho_i}
\end{tikzcd}$$

Note that σ_π is finite surjective morphism. By Lemma 1.25 we have $T_{\pi^*\boldsymbol{\varrho}} = \sigma_\pi^* T_{\boldsymbol{\varrho}}$. By our assumption, for an proper closed subvariety $\Xi \subsetneq S_{\boldsymbol{\varrho}}$, one has $\{T_{\boldsymbol{\varrho}}\}^{\dim \Xi} \cdot \Xi > 0$. Hence for an proper closed subvariety $\Xi \subsetneq S_{\pi^*\boldsymbol{\varrho}}$, one has $\{T_{\pi^*\boldsymbol{\varrho}}\}^{\dim \Xi} \cdot \Xi > 0$. According to Remark 4.16, this implies that

$$\mathrm{Im}[\Lambda^{\dim \Xi}\mathbb{V} \to H^0(\Xi, \Omega_\Xi^{\dim \Xi})] \neq 0.$$

Since $\{T_{\boldsymbol{\varrho}}\}^{\dim S} \cdot S = 0$, it follows that $\{T_{\pi^*\boldsymbol{\varrho}}\}^{\dim S_{\pi^*\boldsymbol{\varrho}}} \cdot S_{\pi^*\boldsymbol{\varrho}} = 0$. This implies that

$$\mathrm{Im}\,[\Lambda^{\dim S_{\pi^*\boldsymbol{\varrho}}}\mathbb{V} \to H^0(S_{\pi^*\boldsymbol{\varrho}}, \Omega^{\dim S_{\pi^*\boldsymbol{\varrho}}}_{S_{\pi^*\boldsymbol{\varrho}}})] = 0.$$

Let r be the generic rank V. According to Remark 4.16, we have $r = \dim S_{\pi^*\boldsymbol{\varrho}} - 1$. By Lemma 4.15, we have $r = \dim_{\mathbb{C}} V$. Therefore, $\mathrm{Im}\,[\Lambda^r V \to H^0(Y, \Omega_Y^r)] \simeq \mathbb{C}$.

Claim 4.18. — *For any non-zero $\eta \in \mathrm{Im}\,[\Lambda^r V \to H^0(Y, \Omega_Y^r)]$, each irreducible component Z' of $(\eta = 0)$ satisfies that $s_{\pi^*\boldsymbol{\varrho}}(Z')$ is a proper subvariety of $S_{\pi^*\boldsymbol{\varrho}}$.*

Proof. — Assume that this is not the case. Let $Z \to Z'$ be a desingularization. Set $V' := \mathrm{Im}\,[V \to H^0(Z, \Omega_Z^1)]$. Denote by r' the generic rank of V'. Then $r' < r$ as Z' is an irreducible component of $(\eta = 0)$. Write $\iota : Z \to Y$ and $g : Z \to X$ for the natural map. Then the Katzarkov-Eyssidieux reduction $s_{g^*\boldsymbol{\varrho}} : Z \to S_{g^*\boldsymbol{\varrho}}$ associated with $g^*\boldsymbol{\varrho}$ is the Stein factorization of the partial Albanese morphism $g_{V'} : Z \to \mathcal{A}_{Z,V'}$. We have the diagram

$$\begin{tikzcd}
Z \arrow[r, "\iota"] \arrow[rr, bend left, "g"] \arrow[d, "s_{g^*\boldsymbol{\varrho}}"] & Y \arrow[r] \arrow[d, "s_{\pi^*\boldsymbol{\varrho}}"] & X \arrow[d, "s_{\boldsymbol{\varrho}}"] \\
S_{g^*\boldsymbol{\varrho}} \arrow[r, "\sigma_\iota"] \arrow[rr, bend right, "\sigma_g"'] & S_{\pi^*\boldsymbol{\varrho}} \arrow[r] & S_{\boldsymbol{\varrho}}
\end{tikzcd}$$

such that σ_ι is a finite *surjective* morphism as we assume that $s_{\pi^*\boldsymbol{\varrho}}(Z') = S_{\pi^*\boldsymbol{\varrho}}$. Let $\Sigma \subsetneq S_{g^*\boldsymbol{\varrho}}$ be a proper closed subvariety. Let $\Sigma' := \sigma_g(\Sigma)$. Since $\{T_{\boldsymbol{\varrho}}\}^{\dim \Sigma'} \cdot \Sigma' > 0$ by our assumption, by Lemma 1.25 $\{T_{g^*\boldsymbol{\varrho}}\}^{\dim \Sigma} \cdot \Sigma > 0$. By Remark 4.16, it follows that the generic rank r' of V' is equal to $\dim S_{g^*\varrho} - 1 = \dim S_{\pi^*\varrho} - 1$. This contradicts with the fact that $r' < r = \dim S_{\pi^*\varrho} - 1$. The claim is proved. $\square$

By the above claim, $s_{\pi^*\varrho}(Z')$ is a proper subvariety of $S_{\pi^*\varrho}$. Therefore, we have $\{T_\varrho\}^{\dim S_{g^*\varrho}} \cdot S_{g^*\varrho} > 0$. Hence for each irreducible component Z of $(\eta = 0)$, $\mathrm{Im}\,[V \to H^0(Z, \Omega_Z^1)]$ is perfect by Remark 4.16 once again. We can apply Theorem 4.12 to conclude that for the holomorphic map $\widetilde{g}_V : Y_V \to V^*$ defined as (4.1), $(Y_V, \widetilde{g}_V^{-1}(t))$ is 1-connected for any $t \in V^*$. For the covering $Y_V \to Y$, we know that $\mathrm{Im}[\pi_1(Y_V) \to \pi_1(Y)]$ contains the derived subgroup $\mathcal{D}\pi_1(Y)$ of $\pi_1(Y)$. Then $\pi^*\varrho_i(\mathrm{Im}\,[\pi_1(Y_V) \to \pi_1(Y)])$ contains $\pi^*\varrho_i(\mathcal{D}\pi_1(Y))$. On the other hand, since $(Y_V, \widetilde{g}_V^{-1}(t))$ is 1-connected for any $t \in V^*$, it follows that $\pi^*\varrho_i(\mathrm{Im}\,[\pi_1(\widetilde{g}_V^{-1}(t)) \to \pi_1(Y)])$ contains $\pi^*\varrho_i(\mathcal{D}\pi_1(Y))$. Note that V is consists of all the spectral forms of $\pi^*\varrho_i$ for all i, hence each $\pi^*\varrho_i$-equivariant harmonic mapping u_i vanishes over each connected component $p^{-1}(\widetilde{g}_V^{-1}(t))$ where $p : \widetilde{Y} \to Y$ is the universal covering. Then $\pi^*\varrho_i(\mathrm{Im}\,[\pi_1(\widetilde{g}_V^{-1}(t)) \to \pi_1(Y)])$ fixes a point P in the Bruhat-Tits building, which implies that it is bounded. Therefore, $\pi^*\varrho_i(\mathcal{D}\pi_1(Y))$ is also bounded. Note that the image of $\pi_1(Y) \to \pi_1(X)$ is a finite index subgroup of $\pi_1(X)$. Hence $\varrho_i(\mathcal{D}\pi_1(X))$ is also bounded for each ϱ_i. The theorem then follows from Lemma 4.8. $\square$

4.4. A factorization theorem. — As an application of Theorem 4.17, we will prove the following factorization theorem which partially generalizes previous theorem by Corlette-Simpson [CS08]. This result is also a warm-up for the proof of Theorem 4.21.

Theorem 4.19. — *Let X be a smooth projective variety and let G be an almost simple algebraic group defined over K. Assume that $\varrho : \pi_1(X) \to G(K)$ is a Zariski dense representation such that for any morphism $f : Z \to X$ from any positive dimensional smooth projective variety Z to X which is birational to the image, the Zariski closure of $f^*\varrho(\pi_1(Z))$ is a semisimple algebraic group. Then after we replace X by a finite étale cover and a birational modification, there is an algebraic fiber space $f : X \to Y$ and a big and Zariski dense representation $\tau : \pi_1(Y) \to G(K)$ such that $f^*\tau = \varrho$. Moreover, $\dim Y \leq \mathrm{rank}_K G$.*

Proof. — We know that there after we replace X by a finite étale cover and a birational modification, there are an algebraic fiber space $f : X \to Y$ over a smooth projective variety Y and a big and Zariski dense representation $\tau : \pi_1(Y) \to G(K)$ such that $f^*\tau = \varrho$. We will prove that $\dim Y \leq \mathrm{rank}_K G$.

Claim 4.20. — *The $(1,1)$-class $\{T_\tau\}$ on S_τ is Kähler, where T_τ is the canonical current on S_τ associated to τ.*

Proof. — By Theorem 1.13, it is equivalent to prove that for any closed subvariety $\Sigma \subset S_\tau$, $\int_\Sigma \{T_\tau\}^{\dim \Sigma} > 0$. We will prove it by induction on $\dim \Sigma$.

Induction. Assume that for every closed subvariety $\Sigma \subset S_\tau$ of dimension $\leq r - 1$, $\{T_\tau\}^{\dim \Sigma} \cdot \Sigma > 0$.

Let Σ be any closed subvariety of S_τ with $\dim \Sigma = r$. Let Z be a desingularization of any irreducible component in $s_\tau^{-1}(\Sigma)$ which is surjective over Σ. Denote by $f : Z \to Y$.

$$
\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "s_{f^*\tau}"] & Y \arrow[d, "s_\tau"] \\
S_{f^*\tau} \arrow[r, "\sigma_f"] & S_\tau
\end{tikzcd}
$$

By Lemma 1.25, σ_f is a finite morphism whose image is Σ and $T_{f^*\tau} = \sigma_f^* T_\tau$.

We first prove the induction for $\dim \Sigma = 1$. In this case $\dim S_{f^*\tau} = 1$. Since the spectral forms associated to $f^*\tau$ are not constant, it follows that $T_{f^*\tau}$ is big. By Lemma 1.25, $\{T_\tau|_\Sigma\}$ is big. Therefore, we prove the induction when $\dim \Sigma = 1$.

Assume now the induction holds for closed subvariety $\Sigma \subset S_\tau$ with $\dim \Sigma \leq r - 1$. Let us deal with the case $\dim \Sigma = r$. By Lemma 1.25 and the induction, we know that for any closed proper positive dimensional subvariety $\Xi \subset S_{f^*\tau}$, we have $\{T_{f^*\tau}\}^{\dim \Xi} \cdot \Xi > 0$. Note that the conditions in Theorem 4.17 for $f^*\tau$ is fulfilled. Therefore, there are two possibilities:

— either $\{T_{f^*\tau}\}^r \cdot S_{f^*\tau} > 0$;
— or the reduction map $s_{f^*\tau} : Z \to S_{f^*\tau}$ coincides with $s_\nu : Z \to S_\nu$, where $\nu : \pi_1(Z) \to (H/\mathcal{D}H)(K)$ is the composition of τ with the group homomorphism $H \to H/\mathcal{D}H$. Here H is the Zariski closure of $f^*\tau$.

If the first case happens, by Lemma 1.25 again we have $\int_\Sigma \{T_\tau\}^{\dim \Sigma} > 0$. we finish the proof of the induction for $\Sigma \subset S_\tau$ with $\dim \Sigma = r$. Assume that the second situation occurs. Since H is assumed to be semisimple, it follows that $H/\mathcal{D}H$ finite. Therefore, ν is bounded and thus $S_{f^*\tau}$ is a point. This contradicts with the fact that $\dim S_{f^*\tau} = \dim \Sigma = r > 0$. Therefore, the second situation cannot occur. We finish the proof of the induction. The claim is proved. □

This claim in particular implies that the *generic rank* r of the multivalued holomorphic 1-forms on Y induced by the differential of harmonic mappings of τ is equal to $\dim S_\tau$.

Since G is almost simple, by [CDY22] we know that the Katzarkov-Eyssidieux reduction map $s_\tau : Y \to S_\tau$ is birational. Therefore $r = \dim Y$. On the other hand, we note that r is less or equal to the dimension of the Bruhat-Tits building $\Delta(G)_K$, which is equal to $\mathrm{rank}_K G$. The theorem is proved. □

4.5. Constructing Kähler classes via representations into non-archimedean fields. — Let X be a smooth projective variety. In this subsection we will prove a more general theorem than Theorem 4.5.

Theorem 4.21. — *Let $\mathfrak{C}$ be absolutely constructible subset of $M_{\mathrm{B}}(X, N)(\mathbb{C})$. Then there is a family of representations $\boldsymbol{\tau} := \{\tau_i : \pi_1(X) \to \mathrm{GL}_N(K_i)\}_{i=1,\ldots,M}$ where K_i are non-archimedean local fields such that*

— *For each $i = 1, \ldots, M$, $[\tau_i] \in \mathfrak{C}(K_i)$;*
— *The reduction map $s_{\boldsymbol{\tau}} : X \to S_{\boldsymbol{\tau}}$ of $\boldsymbol{\tau}$ coincides with $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ defined in Definition 3.1.*
— *For the canonical current $T_{\boldsymbol{\tau}}$ defined over $S_{\mathfrak{C}}$, $\{T_{\boldsymbol{\tau}}\}$ is a Kähler class.*

Proof. — *Step 1.* By Definition 3.1 and Lemma 1.29 there are non-archimedean local fields $L_1, \ldots, L_\ell$ of characteristic zero and reductive representations $\tau_i : \pi_1(X) \to \mathrm{GL}_N(L_i)$ such that $[\tau_i] \in \mathfrak{C}(L_i)$ and $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$ is the Stein factorization of $(s_{\tau_1}, \ldots, s_{\tau_\ell}) : X \to S_{\tau_1} \times \cdots \times S_{\tau_\ell}$. Write $\boldsymbol{\tau} := \{\tau_i\}_{i=1,\ldots,\ell}$. We shall prove that we can add more reductive representations $\tau_{\ell+1}, \ldots, \tau_M$ into non-archimedean local fields L_i with $[\tau_i] \in \mathfrak{C}(L_i)$ for each $i = \ell+1, \ldots, M$ such that $\{T_{\boldsymbol{\tau}'}\}$ over $S_{\mathfrak{C}}$ is Kähler for the new family $\boldsymbol{\tau}' := \{\tau_i : \pi_1(X) \to T(L_i)\}_{i=1,\ldots,M}$.

Step 2. By Theorem 1.13, it suffices to find extra $\tau_{\ell+1}, \ldots, \tau_M$ such that $\{T_{\boldsymbol{\tau}}\}^{\dim \Sigma} \cdot \Sigma > 0$ for every closed subvariety Σ of $S_{\mathfrak{C}}$.

Let $\dim \Sigma = 1$. Let Z be the desingularization of an irreducible component in $s_{\mathfrak{C}}^{-1}(\Sigma)$ which is surjective over Σ. Hence after we reorder $\tau_1, \ldots, \tau_\ell$, one has $s_{\tau_1}(Z) = \Sigma_1$ is a curve. This implies that $\{T_{\tau_1}\} \cdot \Sigma_1 > 0$. Note that $e_{\tau_1} : \Sigma \to \Sigma_1$ is finite. Hence $\{e_{\tau_1}^* T_{\tau_1}\} \cdot \Sigma > 0$. Note that $T_{\boldsymbol{\tau}} \geq e_{\tau_1}^* T_{\tau_1}$. Therefore, $\{T_{\boldsymbol{\tau}}\} \cdot \Sigma > 0$. The case of curves is proved. We now make two inductions of dimension of closed subvarieties in $S_{\mathfrak{C}}$ to prove the theorem.

Induction One. Assume that for every closed subvariety $\Sigma \subset S_{\mathfrak{C}}$ of dimension $\leq r-1$, one can add reductive representations $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(L_i)\}_{i=\ell+1,\ldots,k}$ (depending on Σ) with $[\tau_i] \in \mathfrak{C}(L_i)$ for each $i = \ell+1, \ldots, k$ such that $\{T_{\boldsymbol{\tau}'}\}^{\dim \Sigma} \cdot \Sigma > 0$ for the new family $\boldsymbol{\tau}' := \{\tau_i : \pi_1(X) \to T(L_i)\}_{i=1,\ldots,k}$.

Let $\Sigma \subset S_{\mathfrak{C}}$ be a closed subvariety of dimension r such that $\{T_{\boldsymbol{\tau}}\}^{\dim \Sigma} \cdot \Sigma = 0$. Let $\pi : X^{\text{sp}} \to X$ be a ramified Galois cover which dominates the spectral covers associated to each $\tau_1, \ldots, \tau_\ell$. Pulling back all the spectral forms associated with τ_i to X^{sp}, we obtain a linear space $V \subset H^0(X^{\text{sp}}, \Omega^1_{X^{\text{sp}}})$. We denote by $\mathcal{A}_{X^{\text{sp}}}$ the Albanese variety of X^{sp}. Then $s_{\pi^*\boldsymbol{\tau}} : X^{\text{sp}} \to S_{\pi^*\boldsymbol{\tau}}$ is the Stein factorization of the partial Albanese morphism $X^{\text{sp}} \to \mathcal{A}_{X^{\text{sp}},V}$ associated to V. Then we have a commutative diagram

$$
\begin{tikzcd}
X^{\text{sp}} \arrow[r, "\pi"] \arrow[d, "s_{\pi^*\boldsymbol{\tau}}"] & X \arrow[d, "s_{\mathfrak{C}}"] \\
S_{\pi^*\boldsymbol{\tau}} \arrow[r, "\sigma_\pi"] & S_{\mathfrak{C}}
\end{tikzcd}
$$

with σ_π a finite surjective morphism. Take a closed subvariety $\Sigma' \subset S_{\pi^*\boldsymbol{\tau}}$ which dominates Σ via σ_π. Let Y be the desingularization of the normalization of an irreducible component of $X^{\text{sp}} \times_W \Sigma'$ which dominates Σ'. Consider the pullback representation $\varphi^*\tau_i : \pi_1(Y) \to \mathrm{GL}_N(L_i)$ and the reduction maps $s_{\varphi^*\tau_i} : Y \to S_{\varphi^*\tau_i}$. Then $s_{\varphi^*\boldsymbol{\tau}} : Y \to S_{\varphi^*\boldsymbol{\tau}}$ is the Stein factorization of the partial Albanese morphism associated to $\iota^*V \subset H^0(Y, \Omega^1_Y)$.

$$
\begin{tikzcd}
Y \arrow[r, "\iota"] \arrow[rr, bend left, "\varphi"] \arrow[d, "s_{\varphi^*\boldsymbol{\tau}}"] & X^{\text{sp}} \arrow[r, "\pi"] \arrow[d, "s_{\pi^*\boldsymbol{\tau}}"] & X \arrow[d, "s_{\mathfrak{C}}"] \\
S_{\varphi^*\boldsymbol{\tau}} \arrow[r, "\sigma_\iota"] & S_{\pi^*\boldsymbol{\tau}} \arrow[r, "\sigma_\pi"] & S_{\mathfrak{C}}
\end{tikzcd}
$$

Note that $\sigma_\iota(S_{\varphi^*\boldsymbol{\tau}}) = \Sigma'$. By taking successive hyperplane sections in Y, we can find a morphism $Z' \to Y$ from a smooth projective variety Z' which is birational into the image such that the composition $Z' \to S_{\varphi^*\boldsymbol{\tau}}$ is generically finite surjective morphism.

$$
\begin{tikzcd}
Z' \arrow[r] \arrow[rrr, bend left, "\phi"] \arrow[d, "s_{\phi^*\boldsymbol{\tau}}"] & Y \arrow[r, "\iota"] \arrow[rr, bend left, "\varphi"] \arrow[d, "s_{\varphi^*\boldsymbol{\tau}}"] & X^{\text{sp}} \arrow[r, "\pi"] \arrow[d, "s_{\pi^*\boldsymbol{\tau}}"] & X \arrow[d, "s_{\mathfrak{C}}"] \\
S_{\phi^*\boldsymbol{\tau}} \arrow[r] & S_{\varphi^*\boldsymbol{\tau}} \arrow[r, "\sigma_\iota"] & S_{\pi^*\boldsymbol{\tau}} \arrow[r, "\sigma_\pi"] & S_{\mathfrak{C}}
\end{tikzcd}
$$

Then $S_{\phi^*\boldsymbol{\tau}} \to S_{\varphi^*\boldsymbol{\tau}}$ is a finite surjective morphism by Lemma 1.25. It follows that $s_{\phi^*\boldsymbol{\tau}} : Z' \to S_{\phi^*\boldsymbol{\tau}}$ is a birational morphism. Note that for any reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(K)$, its reduction map $s_\varrho : X \to S_\varrho$ factors through $s_{\mathfrak{C}} : X \to S_{\mathfrak{C}}$. Hence the reduction map $s_{\phi^*\varrho} : Z' \to S_{\phi^*\varrho}$ factors through $s_{\phi^*\boldsymbol{\tau}} : Z' \to S_{\phi^*\boldsymbol{\tau}}$ by Theorem 1.22. We assume that after we add reductive representations into non-archimedean local fields $\tau_{\ell+1}, \ldots, \tau_k$ with $[\tau_i] \in \mathfrak{C}(L_i)$ for each $i = \ell+1, \ldots, k$, the generic rank of the multivalued holomorphic 1-forms on Z' induced by the differential of harmonic mappings of $\{\phi^*\tau_i : \pi_1(Z') \to \mathrm{GL}_N(L_i)\}_{i=1,\ldots,k}$ achieves its *maximum*, which we denoted by m. We take a Galois cover $Z \to Z'$ which dominants all spectral covers of $\phi^*\tau_i$ for $i = 1, \ldots, k$. We replace Z by a desingularization and we pullback all the spectral forms of $\phi^*\tau_i$ for $i = 1, \ldots, k$ to Z to obtain $\mathbb{V} \subset H^0(Z, \Omega^1_Z)$. We still use the same notation $\boldsymbol{\tau}$ to denote the increased family of representation $\{\tau_i\}_{i=1,\ldots,k}$. Note that $s_{\boldsymbol{\tau}}$ always coincides with $s_{\mathfrak{C}}$ if we add $\tau_{\ell+1}, \ldots, \tau_{\ell'}$ with $[\tau_i] \in \mathfrak{C}(L_i)$ for each $i = \ell+1, \ldots, \ell'$. Therefore, the diagram below will stabilize whenever we add such new reductive representations to $\{\tau_1,\ldots,\tau_k\}$.

$$
\begin{tikzcd}
Z \arrow[r, "\pi_0"] \arrow[d, "s_{\psi^*\boldsymbol{\tau}}"] \arrow[rrrr, bend left=50, "\psi"] & Z' \arrow[r] \arrow[d, "s_{\phi^*\boldsymbol{\tau}}"] \arrow[rrr, bend left=40, "\phi"] & Y \arrow[r, "\iota"] \arrow[d, "s_{\varphi^*\boldsymbol{\tau}}"] \arrow[rr, bend left=30, "\varphi"] & X^{\mathrm{sp}} \arrow[r, "\pi"] \arrow[d, "s_{\pi^*\boldsymbol{\tau}}"] & X \arrow[d, "s_{\mathfrak{C}}"] \\
S_{\psi^*\boldsymbol{\tau}} \arrow[r] \arrow[rrrr, bend right=20, "\sigma_\psi"'] & S_{\phi^*\boldsymbol{\tau}} \arrow[r] & S_{\varphi^*\boldsymbol{\tau}} \arrow[r, "\sigma_\iota"] & S_{\pi^*\boldsymbol{\tau}} \arrow[r, "\sigma_\pi"] & S_{\mathfrak{C}}
\end{tikzcd}
\tag{4.2}
$$

Note that $s_{\psi^*\boldsymbol{\tau}}: Z \to S_{\psi^*\boldsymbol{\tau}}$ is the Stein factorization of the partial Albanese morphism $g_{\mathbb{V}}: Z \to \mathcal{A}_{Z,\mathbb{V}}$ associated with $\mathbb{V}$. Note that $s_{\psi^*\boldsymbol{\tau}}$ is birational as $s_{\phi^*\boldsymbol{\tau}}$ is birational. Note that the generic rank of $\mathbb{V}$ is m. Therefore, if $m = \dim Z$, according to Remark 4.16 the current $T_{\psi^*\boldsymbol{\tau}}$ on $S_{\psi^*\boldsymbol{\tau}}$ is big and by the functoriality of the canonical currents in Lemma 1.25, $\{T_{\boldsymbol{\tau}}\}^r \cdot \Sigma > 0$. The induction for subvarieties in $S_{\mathfrak{C}}$ of dimension r is thus proved.

Assume now $m < \dim Z$, which means that the generic rank of $\mathbb{V}$ is less than $\dim Z$. We shall prove that this cannot happen.

Case (1): $m < \dim_{\mathbb{C}} \mathbb{V}$. The proof is closed to Lemma 4.15. By Lemma 4.14 there is $\mathbb{W} \subset \mathbb{V}$ with $\dim_{\mathbb{C}} \mathbb{W} \leq m+1$ such that $\operatorname{Im}[\Lambda^{\dim \mathbb{W}}\mathbb{W} \to H^0(Z, \Omega_Z^{\dim \mathbb{W}})] = 0$, and for any hyperplane $\mathbb{W}' \subsetneq \mathbb{W}$, we always have $\operatorname{Im}[\Lambda^{\dim \mathbb{W}'}\mathbb{W}' \to H^0(Z, \Omega_Z^{\dim \mathbb{W}'})] \neq 0$.

Since we assume that $m < \dim Z$, it follows that $\dim \mathbb{W} \leq \dim Z$. By Theorem 4.13, there is a fibration $p: Z \to B$ with B a projective normal variety with $\dim B = \dim \mathbb{W} - 1 \leq \dim Z - 1$ such that $\mathbb{W} \subset p^*H^0(B, \Omega_B^1)$. Let F be a general fiber of p which is a proper closed subvariety of Z such that F is birational to $F := s_{\psi^*\boldsymbol{\tau}}(F)$ via $s_{\psi^*\boldsymbol{\tau}}$. Since $\mathbb{W} \subset p^*H^0(B, \Omega_B^1)$, the generic rank of $\mathbb{W}$ is equal to $\dim B$, and $\operatorname{Im}[\Lambda^{\dim \mathbb{V}} \to H^0(Z, \Omega_Z^{\dim \mathbb{V}})] = 0$, it implies that

$$\operatorname{Im}[\Lambda^{\dim F}\mathbb{V} \to H^0(F, \Omega_F^{\dim F})] = 0.$$

Therefore, $\{T_{\psi^*\boldsymbol{\tau}}\}^{\dim F} \cdot F = 0$ by Remark 4.16.

By the induction, we can add some new reductive representation $\tau_{k+1},\ldots,\tau_{k'}$ into non-archimedean local fields L_i with $[\tau_i] \in \mathfrak{C}(L_i)$ for each $i = k+1,\ldots,k'$ such that for the new family $\boldsymbol{\tau'} := \{\tau_i\}_{i=1,\ldots,k'}$, one has $\{T_{\boldsymbol{\tau'}}\}^{\dim \sigma_\psi(F)} \cdot \sigma_\psi(F) > 0$. By Lemma 1.25, $\{T_{\psi^*\boldsymbol{\tau'}}\}^{\dim F} \cdot F > 0$.

Note that $s_{\boldsymbol{\tau'}}: X \to S_{\boldsymbol{\tau'}}$ coincides with $s_{\boldsymbol{\tau}}: X \to S_{\boldsymbol{\tau}}$ by our definition of $s_{\mathfrak{C}}: X \to S_{\mathfrak{C}}$. Hence by Lemma 1.25 $s_{\psi^*\boldsymbol{\tau'}}: Z \to S_{\psi^*\boldsymbol{\tau'}}$ coincides with $s_{\psi^*\boldsymbol{\tau}}: Z \to S_{\psi^*\boldsymbol{\tau}}$. Since $\{T_{\psi^*\boldsymbol{\tau'}}\}^{\dim F'} \cdot F' > 0$, we conclude that the rank of multivalued one forms on Z induced by $\psi^*\boldsymbol{\tau'}$ has rank $\dim Z$. It implies that the the rank of multivalued one forms on Z' induced by $\phi^*\boldsymbol{\tau'}$ has rank $\dim Z' = \dim Z$. This contradicts with our assumption that $m < \dim Z$. Hence Case (1) cannot happen. In the next Step we will deal with Case (2) using Theorems 4.5 and 4.17 and show that it can neither happen.

Step 3. Case (2): $m = \dim_{\mathbb{C}} \mathbb{V}$.

Claim 4.22. — *For any reductive representations $\{\tau_i : \pi_1(X) \to \mathrm{GL}_N(L_i')\}_{i=k+1,\ldots,k'}$ with L_i non-archimedean local fields and $[\varrho_i] \in \mathfrak{C}(L_i')$, the new family $\boldsymbol{\tau'} := \{\boldsymbol{\tau}\} \cup \{\tau_i\}_{i=k+1,\ldots,k'}$ satisfies that*

$$CT_{\psi^*\boldsymbol{\tau'}} \geq T_{\psi^*\boldsymbol{\tau}} \geq C^{-1}T_{\psi^*\boldsymbol{\tau'}} \tag{4.3}$$

for some constant $C > 0$.

Proof. — We may replace Z by a Galois cover which dominates the spectral covers of $\psi^*\boldsymbol{\tau}'$. Note that $T_{\psi^*\boldsymbol{\tau}'} \geq T_{\psi^*\boldsymbol{\tau}}$. Note that the rank of multivalued one forms on Z induced by $\psi^*\boldsymbol{\tau}'$ always has rank m by our choice of m. Assume by contradiction that (4.3) does not happen. Then the dimension of global spectral forms $\mathbb{V}'$ induced $\psi^*\boldsymbol{\tau}'$ will be greater than $\dim \mathbb{V}$. We are now in the situation of Case (1), which gives us the contradiction. The claim is proved. □

Claim 4.23. — *For any proper closed subvariety $V \subsetneq \Sigma$ (resp. $V \subsetneq S_{\psi^*\boldsymbol{\tau}}$), one has $\{T_{\boldsymbol{\tau}}\}^{\dim V} \cdot V > 0$ (resp. $\{T_{\psi^*\boldsymbol{\tau}}\}^{\dim V} \cdot V > 0$).*

Proof. — Indeed, by the induction, any proper closed subvariety $V \subsetneq \Sigma$ we can add some new reductive representation $\tau_{k+1}, \ldots, \tau_{k'}$ into non-archimedean local fields L_i with $[\tau_i] \in \mathfrak{C}(L_i)$ for each $i = k+1, \ldots, k'$ such that we have $\{T_{\boldsymbol{\tau}'}\}^{\dim V} \cdot V > 0$. By Lemma 1.25 this implies that $\{T_{\psi^*\boldsymbol{\tau}'}\}^{\dim V'} \cdot V' > 0$ for any closed subvariety $V' \subset S_{\psi^*\boldsymbol{\tau}}$ which dominates V. By Claim 4.22, it follows that $\{T_{\psi^*\boldsymbol{\tau}}\}^{\dim V'} \cdot V' > 0$. The claim follows. □

Let us denote by H_i the Zariski closure of $\psi^*\tau_i(\pi_1(Z))$ for each $i = 1, \ldots, k$, which is a reductive algebraic group over L_i. By [uh], there is some number field k_i and some non-archimedean place v_i of k_i such that $L_i = (k_i)_{v_i}$ and H_i is defined over k_i. Denote $T_i := H_i/\mathcal{D}H_i$. Consider the morphisms of affine k_i-schemes of finite type

$$\begin{array}{ccc} M_{\mathrm{B}}(X,N) & \longrightarrow & M_{\mathrm{B}}(Z,N) \\ & & \uparrow \\ M_{\mathrm{B}}(Z,T_i) & \longleftarrow & M_{\mathrm{B}}(Z,H_i) \end{array} \tag{4.4}$$

Then by Theorem 1.20, $\mathfrak{C} \subset M_{\mathrm{B}}(X,N)(\mathbb{C})$ is transferred via the diagram (4.4) to some absolutely constructible subset $\mathfrak{C}_i$ of $M_{\mathrm{B}}(Z,T_i)$. Consider the reduction map $s_{\mathfrak{C}_i} : Z \to S_{\mathfrak{C}_i}$ defined in Definition 3.1. Denote by $f : Z \to S$ be the Stein factorisation of $s_{\mathfrak{C}_1} \times \cdots \times s_{\mathfrak{C}_k} : Z \to S_{\mathfrak{C}_1} \times \cdots \times S_{\mathfrak{C}_k}$.

Claim 4.24. — *The reduction map $s_{\psi^*\boldsymbol{\tau}} : Z \to S_{\psi^*\boldsymbol{\tau}}$ factors through $Z \xrightarrow{f} S \xrightarrow{q} S_{\psi^*\boldsymbol{\tau}}$.*

Proof. — By Claim 4.23 the conditions in Theorem 4.17 are fulfilled for $\psi^*\boldsymbol{\tau}$. Since we assume that $m < \dim Z$, which means that the generic rank of $\mathbb{V}$ is less than $\dim Z$. It implies that $\{T_{\psi^*\boldsymbol{\tau}}\}^{\dim S_{\psi^*\boldsymbol{\tau}}} \cdot S_{\psi^*\boldsymbol{\tau}} = 0$. Hence the second possibility in Theorem 4.17 happens and thus we conclude that the reduction map $s_{\sigma_i} : Z \to S_{\sigma_i}$ coincides with $s_{\psi^*\tau_i} : Z \to S_{\psi^*\tau_i}$ where $\sigma_i : \pi_1(Z) \to T_i(L_i)$ is the composition of $\psi^*\tau_i : \pi_1(Z) \to H_i(L_i)$ with the group homomorphism $H_i \to T_i$. By (4.4) and the definition of $\mathfrak{C}_i$, $[\sigma_i] \in \mathfrak{C}_i(L_i)$. Therefore, s_{σ_i} factors through $s_{\mathfrak{C}_i}$. Since $s_{\sigma_i} : Z \to S_{\sigma_i}$ coincides with $s_{\psi^*\tau_i} : Z \to S_{\psi^*\tau_i}$, it follows that $s_{\psi^*\boldsymbol{\tau}}$ factors through $Z \xrightarrow{f} S \xrightarrow{q} S_{\psi^*\boldsymbol{\tau}}$. □

Since T_i are all algebraic tori defined over number fields k_i, we apply Theorem 4.5 to conclude that there exists a family of reductive representations $\boldsymbol{\varrho}_i := \{\varrho_{ij} : \pi_1(Z) \to T_i(K_{ij})\}_{j=1,\ldots,n_i}$ with K_{ij} non-archimedean local field such that

(1) For each $i = 1, \ldots, k; j = 1, \ldots, n_i$, $[\varrho_{ij}] \in \mathfrak{C}_i(K_{ij})$;
(2) The reduction map $s_{\boldsymbol{\varrho}_i} : Z \to S_{\boldsymbol{\varrho}_i}$ of $\boldsymbol{\tau}_i$ coincides with $s_{\mathfrak{C}_i} : Z \to S_{\mathfrak{C}_i}$;
(3) for the canonical current $T_{\boldsymbol{\varrho}_i}$ over $S_{\mathfrak{C}_i}$ associated with $\boldsymbol{\varrho}_i$, $\{T_{\boldsymbol{\varrho}_i}\}$ is a Kähler class.

By the definition of $\mathfrak{C}_i$, there exist a finite extension F_{ij} of K_{ij} and reductive representations $\{\delta_{ij} : \pi_1(X) \to \mathrm{GL}_N(F_{ij})\}_{j=1,\ldots,n_i}$ such that

(a) For each $i = 1, \ldots, k; j = 1, \ldots, n_i$, $[\delta_{ij}] \in \mathfrak{C}(F_{ij})$;
(b) the Zariski closure of $\psi^*\delta_{ij} : \pi_1(Z) \to \mathrm{GL}_N(F_{ij})$ is contained in H_i;
(c) $[\eta_{ij}] = [\varrho_{ij}] \in M_{\mathrm{B}}(Z,T_i)(F_{ij})$, where $\eta_{ij} : \pi_1(Z) \to T_i(F_{ij})$ is the composition of $\psi^*\delta_{ij} : \pi_1(Z) \to H_i(F_{ij})$ with the group homomorphism $H_i \to T_i$.

Therefore, η_{ij} is conjugate to ϱ_{ij} and thus their reduction map coincides. It follows that the canonical currents $T_{\eta_{ij}}$ coincides with $T_{\varrho_{ij}}$. Let R_i be the radical of H_i. Write $\eta'_{ij}:\pi_1(Z)\to(H_i/R_i)(F_{ij})$ to be the composition of $\psi^*\delta_{ij}:\pi_1(Z)\to H_i(F_{ij})$ with the homomorphism $H_i\to H_i/R_i$. Note that $H_i\to T_i\times H_i/R_i$ is an isogeny. It follows that the reduction map $s_{\psi^*\delta_{ij}}$ is the Stein factorization of $s_{\eta_{ij}}\times s_{\eta'_{ij}}:Z\to S_{\eta_{ij}}\times S_{\eta'_{ij}}$. Therefore, the reduction map $s_{\eta_{ij}}:Z\to S_{\eta_{ij}}$ factors through the reduction map $s_{\psi^*\delta_{ij}}:Z\to S_{\psi^*\delta_{ij}}$ with the finite morphism $q_{ij}:S_{\psi^*\delta_{ij}}\to S_{\eta_{ij}}$. Moreover, By Definition 1.24, one can see that

$$q_{ij}^*T_{\varrho_{ij}}=q_{ij}^*T_{\eta_{ij}}\leq T_{\psi^*\delta_{ij}}. \tag{4.5}$$

Consider the family of representations $\boldsymbol{\delta}:=\{\delta_{ij}:\pi_1(X)\to \mathrm{GL}_N(F_{ij})\}_{i=1,\ldots,k;j=1,\ldots,n_i}$. By Items (2) and (c)the Stein factorization $f:Z\to S$ of $Z\to S_{\mathfrak{C}_1}\times\cdots\times S_{\mathfrak{C}_k}$ factors through the reduction map $s_{\psi^*\boldsymbol{\delta}}:Z\to S_{\psi^*\boldsymbol{\delta}}$. By Claim 4.24, $f:Z\to S$ coincides with $s_{\psi^*\boldsymbol{\delta}}:Z\to S_{\psi^*\boldsymbol{\delta}}$. Let $e_i:S\to S_{\mathfrak{C}_i}=S_{\boldsymbol{\varrho}_i}$ be the natural map. Note that $e_1\times\cdots\times e_k:S\to S_{\mathfrak{C}_1}\times\cdots\times S_{\mathfrak{C}_k}$ is finite. By Items (2) and (3), $\{\sum_{i=1}^k e_i^*T_{\boldsymbol{\varrho}_i}\}$ is Kähler on $S=S_{\mathfrak{C}}$. By (4.5), we conclude that $\{T_{\psi^*\boldsymbol{\delta}}\}$ is Kähler on $S_{\mathfrak{C}}$. According to Remark 4.16, it implies that the generic rank m of the multivalued holomorphic 1-forms on Z' induced by the differential of harmonic mappings associated with $\{\phi^*\delta_{ij}:\pi_1(Z')\to \mathrm{GL}_N(F_{ij})\}_{i=1,\ldots,k;j=1,\ldots,n_i}$ is equal to $\dim Z$. This contradicts with our assumption that $m<\dim Z$. Hence Case (2) can neither happen. We prove Induction one.

Step 4. We now prove the theorem by another induction.

Induction Two. Assume that for every closed subvariety $\Sigma\subset S_{\mathfrak{C}}$ of dimension $\leq r-1$, one can add $\tau_{\ell+1},\ldots,\tau_p$ (depending on Σ) with $[\tau_i]\in\mathfrak{C}(L_i)$ for each $i=\ell+1,\ldots,p$ such that for every closed subvariety $\Xi\subset\Sigma$, one has $\{T_{\boldsymbol{\tau}}\}^{\dim\Xi}\cdot\Xi>0$, where $\boldsymbol{\tau}:=\{\tau_i\}_{i=1,\ldots,p}$.

Obviously, this induction is the same as Induction one for $\dim\Sigma=1$ and thus it holds in this case. Let $\Sigma\subset S_{\mathfrak{C}}$ be a closed subvariety of dimension r. We shall prove that the induction holds for such Σ.

By Induction One, one can add reductive representations $\{\tau_i:\pi_1(X)\to \mathrm{GL}_N(L_i)\}_{i=\ell+1,\ldots,k}$ (depending on Σ) with $[\tau_i]\in\mathfrak{C}(L_i)$ for each $i=\ell+1,\ldots,k$ such that $\{T_{\boldsymbol{\tau}'}\}^{\dim\Sigma}\cdot\Sigma>0$ for the new family $\boldsymbol{\tau}':=\{\tau_i:\pi_1(X)\to T(L_i)\}_{i=1,\ldots,k}$. We construct $\psi:Z\to X$ a diagram as (4.2). Then $\{T_{\psi^*\boldsymbol{\tau}'}\}$ is a big class on $S_{\psi^*\boldsymbol{\tau}'}$ by Lemma 1.25. We may replace Z by a Galois cover which dominates the spectral covers of $\psi^*\boldsymbol{\tau}'$. Let $V\subset H^0(Z,\Omega_Z^1)$ be the subspace generated by all spectral one forms induced by $\psi^*\boldsymbol{\tau}'$. Note that there is a subspace $\mathbb{V}\subset H^0(S_{\psi^*\boldsymbol{\tau}'},\Omega^1_{S_{\psi^*\boldsymbol{\tau}'}})$ such that $s_{\psi^*\boldsymbol{\tau}'}^*\mathbb{V}=V$. By Remark 4.16,

$$\mathrm{Im}\,[\Lambda^{\dim S_{\psi^*\boldsymbol{\tau}'}}\mathbb{V}\to H^0(S_{\psi^*\boldsymbol{\tau}'},\Omega^{\dim S_{\psi^*\boldsymbol{\tau}'}}_{S_{\psi^*\boldsymbol{\tau}'}})]\neq 0.$$

Pick some non-zero $\eta\in\mathrm{Im}\,[\Lambda^{\dim S_{\psi^*\boldsymbol{\tau}'}}\mathbb{V}\to H^0(S_{\psi^*\boldsymbol{\tau}'},\Omega^{\dim S_{\psi^*\boldsymbol{\tau}'}}_{S_{\psi^*\boldsymbol{\tau}'}})]$. Let $Z_1,\ldots,Z_c$ be all irreducible components of $(\eta=0)$. Denote by $W_i:=\sigma_\psi(Z_i)$. Since the image of $\sigma_\psi: S_{\psi^*\boldsymbol{\tau}'}\to S_{\mathfrak{C}}$ is Σ, W_i's are proper closed subvarieties of Σ. Let Ξ be a proper closed subvariety in Σ such that $\{T_{\boldsymbol{\tau}}\}^{\dim\Xi}\cdot\Xi=0$. We can take a closed proper subvariety Ξ' such that $\sigma_\psi(\Sigma')=\Sigma$. Then by Lemma 1.25, $\{T_{\psi^*\boldsymbol{\tau}'}\}^{\dim\Xi'}\cdot\Xi'=0$. Therefore, Ξ' must be contained in some Z_i. It follows that Ξ is contained in some W_i.

Since $\dim W_i\leq r-1$, by Induction Two we can add more reductive representations $\tau_{k+1},\ldots,\tau_{k'}$ with $[\tau_i]\in\mathfrak{C}(L_i)$ for each $i=k+1,\ldots,k'$ such that for the new family $\boldsymbol{\tau}'':=\{\tau_i\}_{i=1,\ldots,k'}$, one has $\{T_{\boldsymbol{\tau}''}\}^{\dim\Xi}\cdot\Xi>0$ for all closed subvarieties Ξ contained in $\cup_{i=1}^c W_i$. Since $T_{\boldsymbol{\tau}''}\geq T_{\boldsymbol{\tau}'}$, it follows that $\{T_{\boldsymbol{\tau}''}\}^{\dim\Xi}\cdot\Xi>0$ for all closed subvarieties Ξ contained in Σ. Induction Two is proved.

Step 5. We now apply Induction Two to $S_{\mathfrak{C}}$. Then we can add reductive representations $\tau_{\ell+1},\ldots,\tau_M$ with $[\tau_i]\in\mathfrak{C}(L_i)$ for each $i=\ell+1,\ldots,M$ such that $\{T_{\boldsymbol{\tau}'}\}^{\dim\Sigma}\cdot\Sigma>0$ for every closed subvariety Z of $S_{\mathfrak{C}}$, where $\boldsymbol{\tau}'=\{\tau_i:\pi_1(X)\to \mathrm{GL}_N(L_i)\}_{i=1,\ldots,M}$. Hence $\{T_{\boldsymbol{\tau}'}\}$ is Kähler by Theorem 1.13. We complete the proof of the theorem. □

4.6. Holomorphic convexity associated with absolutely constructible subsets. — In this subsection we will prove Theorem B. We shall use the notations and results proven in § 3.3 and Theorem 3.20 without recalling the details.

Theorem 4.25. — *Let X be a smooth projective variety. Let $\mathfrak{C}$ be an absolutely constructible subset of $M_{\mathrm{B}}(X,N)(\mathbb{C})$ defined in Definition 1.17. Assume that $\mathfrak{C}$ is defined on $\mathbb{Q}$. Let $\pi:\widetilde{X}_{\mathfrak{C}}\to X$ be the covering corresponding to the group $\cap_{\varrho}\ker\varrho\subset\pi_1(X)$ where $\varrho:\pi_1(X)\to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations such that $[\varrho]\in\mathfrak{C}(\mathbb{C})$. Then $\widetilde{X}_{\mathfrak{C}}$ is holomorphically convex. In particular, if $\pi_1(X)$ is a subgroup of $\mathrm{GL}_N(\mathbb{C})$ whose Zariski closure is reductive, then $\widetilde{X}_{\mathfrak{C}}$ is holomorphically convex.*

Proof. — Let $H:=\cap_{\varrho}\ker\varrho\cap\sigma$, where σ is the $\mathbb{C}$-VHS defined in Proposition 3.12 and $\varrho:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representation such that $[\varrho]\in\mathfrak{C}$. Denote by $\widetilde{X}_H:=\widetilde{X}/H$. Let $\mathscr{D}$ be the period domain associated to the $\mathbb{C}$-VHS σ defined in Proposition 3.12 and let $p:\widetilde{X}_H\to\mathscr{D}$ be the period mapping. By (3.5), $H=\cap_{\varrho}\ker\varrho$, where $\varrho:\pi_1(X)\to\mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representation such that $[\varrho]\in\mathfrak{C}$. Therefore, $\widetilde{X}_{\mathfrak{C}}=\widetilde{X}_H$.

Consider the product

$$\Psi=s_{\mathfrak{C}}\circ\pi_H\times p:\widetilde{X}_H\to S_{\mathfrak{C}}\times\mathscr{D}$$

where $p:\widetilde{X}_H\to\mathscr{D}$ is the period mapping of σ. Recall that Ψ factors through a proper surjective fibration $\mathrm{sh}_H:\widetilde{X}_H\to\widetilde{S}_H$. Moreover, there is a properly discontinuous action of $\pi_1(X)/H$ on $\widetilde{S}_H$ such that sh_H is equivariant with respect to this action. Write $g:\widetilde{S}_H\to S_{\mathfrak{C}}\times\mathscr{D}$ to be the induced holomorphic map. Denote by $\phi:\widetilde{S}_H\to\mathscr{D}$ the composition of g and the projection map $S_{\mathfrak{C}}\times\mathscr{D}\to\mathscr{D}$. Since the period mapping p is horizontal, and sh_H is surjective, it follows that ϕ is also horizontal.

Recall that in Lemma 3.28 we prove that there is a finite index normal subgroup N of $\pi_1(X)/H$ and a homomorphism $\nu:N\to\mathrm{Aut}(\widetilde{S}_H)$ such that $\mathrm{sh}_H:\widetilde{X}_H\to\widetilde{S}_H$ is ν-equivariant and $\nu(N)$ acts on $\widetilde{S}_H$ properly discontinuous and without fixed point. Let $Y:=\widetilde{X}_H/N$. Moreover, $c:Y\to X$ is a finite Galois étale cover and N gives rise to a proper surjective fibration $Y\to\widetilde{S}_H/\nu(N)$ between compact normal complex spaces. Write $W:=\widetilde{S}_H/\nu(N)$. Then $\widetilde{S}_H\to W$ is a topological Galois unramified covering. Recall that the canonical bundle $K_{\mathscr{D}}$ on the period domain $\mathscr{D}$ can be endowed with a G_0-invariant smooth metric $h_{\mathscr{D}}$ whose curvature is strictly positive-definite in the horizontal direction. As $\phi:\widetilde{S}_H\to\mathscr{D}$ is $\nu(N)$-equivariant, it follows that $\phi^*K_{\mathscr{D}}$ descends to a line bundle on the quotient $W:=\widetilde{S}_H/\nu(N)$, denoted by L_G. The smooth metric $h_{\mathscr{D}}$ induces a smooth metric on L_G whose curvature form is denoted by T. Let $x\in W$ be a smooth point of W and let $v\in T_{\widetilde{S}_H,x}$. Then $|v|^2_{\omega}>0$ if $d\phi(v)\neq 0$.

We fix a reference point x_0 on $\widetilde{S}_H$. Define $\phi_0:=2d^2_{\mathscr{D}}(\phi(x),\phi(x_0))$ where $d_{\mathscr{D}}:\mathscr{D}\times\mathscr{D}\to\mathbb{R}_{\geq 0}$ is the distance function on the period domain $\mathscr{D}$. By [Eys04, Theorem 3.3.2], we have

$$\mathrm{dd}^{\mathrm{c}}\phi_0\geq\omega=q^*T. \tag{4.6}$$

where we $q:\widetilde{S}_H\to\widetilde{S}_H/\nu(N)$ denotes the quotient map.

We now apply Theorem 4.21 to find a family of representations $\boldsymbol{\tau}:=\{\tau_i:\pi_1(X)\to\mathrm{GL}_N(K_i)\}_{i=1,\ldots,m}$ where K_i are non-archimedean local fields such that

— For each $i=1,\ldots,m$, $[\tau_i]\in\mathfrak{C}(K_i)$;
— The reduction map $s_{\boldsymbol{\tau}}:X\to S_{\boldsymbol{\tau}}$ of $\boldsymbol{\tau}$ coincides with $s_{\mathfrak{C}}$.
— For the canonical current $T_{\boldsymbol{\tau}}$ defined over $S_{\mathfrak{C}}$, $\{T_{\boldsymbol{\tau}}\}$ is a Kähler class.

Consider

$$
\begin{tikzcd}
\widetilde{X}_H \arrow[r] \arrow[d, "\mathrm{sh}_H"] & Y \arrow[r, "c"] \arrow[d] & X \arrow[d, "s_{\mathfrak{C}}"] \arrow[dr, "s_{\boldsymbol{\tau}}"] \arrow[drr, bend left, "s_{\tau_i}"] & & \\
\widetilde{S}_H \arrow[r, "q"] \arrow[rr, bend right, "r"'] \arrow[rrrr, bend right=40, "r_i"'] & W \arrow[r, "f"] & S_{\mathfrak{C}} \arrow[r, equal] & S_{\boldsymbol{\tau}} \arrow[r, "e_i"] & S_{\tau_i}
\end{tikzcd}
$$

Note that p is a finite surjective morphism.

We fix a reference point x_0 on $\widetilde{S}_H$. For each $i = 1, \ldots, m$, let $u_i : \widetilde{X}_H \to \Delta(\mathrm{GL}_N)_{K_i}$ be the τ_i-equivariant harmonic mapping from $\widetilde{X}_H$ to the Bruhat-Tits building of $\mathrm{GL}_N(K_i)$ whose existence was ensured by a theorem of Gromov-Schoen [GS92]. Then the function $\tilde{\phi}_i(x) := 2d_i^2(u_i(x), u_i(x_0))$ defined over $\widetilde{X}_H$ is locally Lipschitz, where $d_i : \Delta(\mathrm{GL}_N)_{K_i} \times \Delta(\mathrm{GL}_N)_{K_i} \to \mathbb{R}_{\geq 0}$ is the distance function on the Bruhat-Tits building. By Proposition 1.26, it induces a continuous psh functions $\{\phi_i : \widetilde{S}_H \to \mathbb{R}_{\geq 0}\}_{i=1,\ldots,m}$ such that $\mathrm{dd^c}\phi_i \geq r_i^* T_{\tau_i}$ for each i. By the definition of $T_{\boldsymbol{\tau}}$, we have

$$
\mathrm{dd^c} \sum_{i=1}^{m} \phi_i \geq r^* T_{\boldsymbol{\tau}}. \tag{4.7}
$$

Therefore, putting (4.6) and (4.7) together we obtain

$$
\mathrm{dd^c} \sum_{i=0}^{m} \phi_i \geq q^*(f^* T_{\boldsymbol{\tau}} + T). \tag{4.8}
$$

As f is a finite surjective morphism, $\{f^* T_{\boldsymbol{\tau}}\}$ is also Kähler by Theorem 1.13.

By Claim 3.31, we know that $g : \widetilde{S}_H \to S_{\mathfrak{C}} \times \mathscr{D}$ has discrete fibers. Since T is induced by the curvature form of $(K_{\mathscr{D}}, h_{\mathscr{D}})$, and $\phi : \widetilde{S}_H \to \mathscr{D}$ is horizontal, we can prove that for every irreducible positive dimensional closed subvariety Z of W, $f^* T_{\boldsymbol{\tau}} + T$ is strictly positive at general smooth points of Z. Therefore,

$$
\{f^* T_{\boldsymbol{\tau}} + T\}^{\dim Z} \cdot Z = \int_Z (f^* T_{\boldsymbol{\tau}} + T)^{\dim Z} > 0.
$$

Recall that W is projective by the proof of Claim 3.32. We utilize Theorem 1.13 to conclude that $\{f^* T_{\boldsymbol{\tau}} + T\}$ is Kähler.

Given that $\widetilde{S}_H \to W$ represents a topological Galois unramified cover, we can apply Proposition 1.14 in conjunction with (4.8) to deduce that $\widetilde{S}_H$ is a Stein manifold. Furthermore, since $\widetilde{X}_H \to \widetilde{S}_H$ is a proper surjective holomorphic fibration, the holomorphic convexity of $\widetilde{X}_H$ follows from the Cartan-Remmert theorem. Ultimately, the theorem is established by noting that $\widetilde{X}_H = \widetilde{X}_{\mathfrak{C}}$. □

4.7. Universal covering is Stein. — We shall use the notations in the proof of Theorem 4.25 without recalling their definitions.

Theorem 4.26. — *Let X be a smooth projective variety. Consider an absolutely constructible subset $\mathfrak{C}$ of $M_{\mathrm{B}}(X, \mathrm{GL}_N(\mathbb{C}))$ as defined in Definition 1.17. We further assume that $\mathfrak{C}$ is defined over $\mathbb{Q}$. If $\mathfrak{C}$ is considered to be* large, *meaning that for any closed subvariety Z of X, there exists a reductive representation $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in \mathfrak{C}$ and $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)])$ is infinite, then all intermediate coverings between $\widetilde{X}$ and $\widetilde{X}_{\mathfrak{C}}$ of X are Stein manifolds.*

Proof. — Note that $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$ is a proper holomorphic surjective fibration.

Claim 4.27. — sh_H *is biholomorphic.*

Proof. — Assume that there exists a positive-dimensional compact subvariety Z of $\widetilde{X}_H$ which is contained in some fiber of sh_H. Consider $W := \pi_H(Z)$ which is a compact positive-dimensional irreducible subvariety of X. Therefore, $\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(W^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(W^{\mathrm{norm}})$. By the definition of $\widetilde{X}_H$, for any reductive $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ with $[\varrho] \in \mathfrak{C}(\mathbb{C})$, we have $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)]) = \{1\}$. Therefore, $\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(X)]) = \{1\}$ is finite. This contradicts with out assumption that $\mathfrak{C}$ is large. Hence, sh_H is a one-to-one proper holomorphic map of complex normal spaces. Consequently, it is biholomorphic. □

By the proof of Theorem 4.25, there exist

— a topological Galois unramified covering $q : \widetilde{X}_H = \widetilde{S}_H \to W$, where W is a projective normal variety;

— a positive $(1,1)$-current with continuous potential $f^*T_{\boldsymbol{\tau}} + T$ over W such that $\{f^*T_{\boldsymbol{\tau}} + T\}$ is Kähler;

— a continuous semi-positive plurisubharmonic function $\sum_{i=0}^{m} \phi_i$ on $\widetilde{S}_H$ such that we have

$$\mathrm{dd^c} \sum_{i=0}^{m} \phi_i \geq q^*(f^*T_{\boldsymbol{\tau}} + T). \tag{4.9}$$

Let $p : \widetilde{X}' \to \widetilde{X}_H$ be the intermediate Galois covering of X between $\widetilde{X} \to \widetilde{X}_H$. By (4.6) we have

$$\mathrm{dd^c} \sum_{i=0}^{m} p^*\phi_i \geq (q \circ p)^*(f^*T_{\boldsymbol{\tau}} + T). \tag{4.10}$$

We apply Proposition 1.14 to conclude that $\widetilde{X}'$ is Stein. □

Appendix A. Shafarevich conjecture for projective normal varieties

Ya Deng, Ludmil Katzarkov [(a)] & Katsutoshi Yamanoi

In this appendix, we aim to extend Theorems 4.25 and 4.26 to include singular normal varieties, and thus completing the proofs of Theorems B and C.

A.1. Absolutely constructible subset (II). — Let X be a projective normal variety. Following the recent work of Lerer [Ler22], we can also define absolutely constructible subsets in the character variety $M_{\mathrm{B}}(X,N) := M_{\mathrm{B}}(\pi_1(X), \mathrm{GL}_N)$.

Definition A.1. — Let X be a normal projective variety, $\mu : Y \to X$ be a resolution of singularities, and $\iota : M_{\mathrm{B}}(X,N) \hookrightarrow M_{\mathrm{B}}(Y,N)$ be the embedding. A subset $\mathfrak{C} \subset M_{\mathrm{B}}(X,N)(\mathbb{C})$ is called *absolutely constructible* if $\iota(\mathfrak{C})$ is an absolutely constructible subset of $M_{\mathrm{B}}(Y,N)$ in the sense of Definition 1.17.

Note that the above definition does not depend on the choice of the resolution of singularities (cf. [Ler22, Lemma 2.7]). Moreover, we have the following result.

***Proposition A.2* ([Ler22, Proposition 2.8]).** — *Let X be a normal projective variety. Then $M_{\mathrm{B}}(X,N)$ is absolutely constructible in the sense of Definition A.1.*

This result holds significant importance, as it provides a fundamental example of absolutely constructible subsets for projective normal varieties. It is worth noting that in [Ler22, Proposition 2.8], it is explicitly stated that $\iota(M_{\mathrm{B}}(X,N))$ is $U(1)$-invariant, with ι defined in Definition A.1. However, it should be emphasized that the proof can be easily adapted to show $\mathbb{C}^*$-invariance, similar to the approach used in the proof of Proposition 3.35.

a. E-mail: **l.katzarkov@miami.edu**, University of Miami, Coral Gables, FL, USA; Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria; NRU HSE, Moscow, Russia

A.2. Reductive Shafarevich conjecture for normal projective varieties. —

Theorem A.3. — *Let Y be a projective normal variety. Let $\mathfrak{C}$ be an absolutely constructible subset of $M_{\mathrm{B}}(Y,N)(\mathbb{C})$, defined on $\mathbb{Q}$ (e.g. $\mathfrak{C} = M_{\mathrm{B}}(Y,N)$). Consider the covering $\pi : \widetilde{Y}_{\mathfrak{C}} \to Y$ corresponding to the subgroup $\cap_{\varrho} \ker \varrho$ of $\pi_1(Y)$, where $\varrho : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations such that $[\varrho] \in \mathfrak{C}$. Then the complex space $\widetilde{Y}_{\mathfrak{C}}$ is holomorphically convex. In particular,*

- *The covering corresponding to the intersection of the kernels of all reductive representations of $\pi_1(Y)$ in $\mathrm{GL}_N(\mathbb{C})$ is holomorphically convex;*
- *if $\pi_1(Y)$ is a subgroup of $\mathrm{GL}_N(\mathbb{C})$ whose Zariski closure is reductive, then the universal covering of Y is holomorphically convex.*

Proof. — Let $\mu : X \to Y$ be any desingularization. Let $j : M_{\mathrm{B}}(Y,N) \hookrightarrow M_{\mathrm{B}}(X,N)$ the closed immersion induced by μ, which is a morphism of affine $\mathbb{Q}$-schemes of finite type . Then by Definition A.1, $j(\mathfrak{C})$ is an absolutely constructible in the sense of Definition 1.17. Since $\mathfrak{C}$ is defined on $\mathbb{Q}$, so is $j(\mathfrak{C})$. We shall use the notations in Theorem 3.20. Let $\widetilde{X}_H$ be the covering associated with the subgroup $H := \cap_{\varrho} \ker \varrho$ of $\pi_1(X)$ where $\varrho : \pi_1(X) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations such that $[\varrho] \in j(\mathfrak{C})(\mathbb{C})$. In other words, $H := \cap_{\tau} \ker \mu^*\tau$ where $\tau : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations such that $[\tau] \in \mathfrak{C}(\mathbb{C})$. Denote by $H_0 := \cap_{\tau} \ker \tau$ where $\tau : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ ranges over all reductive representations such that $[\varrho] \in \mathfrak{C}(\mathbb{C})$. Therefore, $H = (\mu_*)^{-1}(H_0)$, where $\mu_* : \pi_1(X) \to \pi_1(Y)$ is a surjective homeomorphism as Y is normal. Therefore, the natural homeomorphism $\pi_1(X)/H \to \pi_1(Y)/H_0$ is an isomorphism. Then $\widetilde{X}_{\mathfrak{C}} = \widetilde{X}/H$ and $\widetilde{Y}_H := \widetilde{Y}/H_0$ where $\widetilde{X}$ (resp. $\widetilde{Y}$) is the universal covering of X (resp. X). It induces a lift $p : \widetilde{X}_H \to \widetilde{Y}_{\mathfrak{C}}$ such that

$$\begin{tikzcd} \widetilde{X}_H \arrow[r, "\pi_H"] \arrow[d, "p"] & X \arrow[d, "\mu"] \\ \widetilde{Y}_{\mathfrak{C}} \arrow[r, "\pi"] & Y \end{tikzcd}$$

Claim A.4. — *$p : \widetilde{X}_H \to \widetilde{Y}_{\mathfrak{C}}$ is a proper surjective holomorphic fibration with connected fibers.*

Proof. — Note that $\mathrm{Aut}(\widetilde{X}_H/X) = \pi_1(X)/H \simeq \pi_1(Y)/H_0 = \mathrm{Aut}(\widetilde{Y}_{\mathfrak{C}}/Y)$. Therefore, $\widetilde{X}_H$ is the base change $\widetilde{Y}_{\mathfrak{C}} \times_Y X$. Note that each fiber of μ is connected as Y is normal. It follows that each fiber of p is connected. The claim is proved. □

By Theorem 3.20, we know that there exist a proper surjective holomorphic fibration $\mathrm{sh}_H : \widetilde{X}_H \to \widetilde{S}_H$ such that $\widetilde{S}_H$ is a Stein space. Therefore, for each connected compact subvariety $Z \subset \widetilde{X}_H$, $\mathrm{sh}_H(Z)$ is a point. By Claim A.4, it follows that each fiber of p is compact and connected, and thus is contracted by sh_H. Therefore, sh_H factors through a proper surjective fibration $f : \widetilde{X}_{\mathfrak{C}} \to \widetilde{S}_H$:

$$\begin{tikzcd} \widetilde{X}_H \arrow[d, "p"] \arrow[dr, "\mathrm{sh}_H"] & \\ \widetilde{Y}_{\mathfrak{C}} \arrow[r, "f"] & \widetilde{S}_H \end{tikzcd}$$

Therefore, f is a proper surjective holomorphic fibration over a Stein space. By the Cartan-Remmert theorem, $\widetilde{Y}_{\mathfrak{C}}$ is holomorphically convex.

If we define $\mathfrak{C}$ as $M_{\mathrm{B}}(Y,N)$, then according to Proposition A.2, $\mathfrak{C}$ is also absolutely constructible. As a result, the last two claims can be deduced. Thus, the theorem is proven. □

Theorem A.5. — *Let Y be a projective normal variety. Let $\mathfrak{C}$ be an absolutely constructible subset of $M_{\mathrm{B}}(Y,N)(\mathbb{C})$, defined on $\mathbb{Q}$ (e.g. $\mathfrak{C} = M_{\mathrm{B}}(Y,N)$). Let $\mathfrak{C}(\mathbb{C})$ be* large *in the sense that for any closed positive dimensional subvariety Z of Y, there exists a reductive representation $\varrho : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ such that $[\varrho] \in \mathfrak{C}(\mathbb{C})$ and $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(Y)])$ is infinite. Then all intermediate Galois coverings of Y between $\widetilde{Y}$ and $\widetilde{Y}_{\mathfrak{C}}$ are Stein spaces. Here $\widetilde{Y}$ denotes the universal covering of Y.*

Proof. — Let $\mu : X \to Y$ be any desingularization. In the following, we will use the same notations as in the proof of Theorem A.3 without explicitly recalling their definitions. Recall that we have constructed three proper surjective holomorphic fibrations p, f, and sh_H satisfying the following commutative diagram:

$$\begin{tikzcd} & \widetilde{X}_H \arrow[r, "\pi_H"] \arrow[d, "p"] \arrow[dl, "\mathrm{sh}_H"'] & X \arrow[d, "\mu"] \\ \widetilde{S}_H & \widetilde{Y}_{\mathfrak{C}} \arrow[l, "f"] \arrow[r, "\pi"] & Y \end{tikzcd}$$

Claim A.6. — *$f : \widetilde{Y}_{\mathfrak{C}} \to \widetilde{S}_H$ is a biholomorphism.*

The proof follows a similar argument to that of Claim 4.27. For the sake of completeness, we will provide it here.

Proof of Claim A.6. — As each fibers of f is compact and connected, it suffices to prove that there are no compact positive dimensional subvarieties Z of $\widetilde{Y}_{\mathfrak{C}}$ such that $f(Z)$ is a point. Let us assume, for the sake of contradiction, that such a Z exists. Consider $W := \pi(Z)$ which is a compact positive-dimensional irreducible subvariety of Y. Therefore, $\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(W^{\mathrm{norm}})]$ is a finite index subgroup of $\pi_1(W^{\mathrm{norm}})$. By the definition of $\widetilde{Y}_{\mathfrak{C}}$, for any reductive $\varrho : \pi_1(Y) \to \mathrm{GL}_N(\mathbb{C})$ with $[\varrho] \in \mathfrak{C}(\mathbb{C})$, we have $\varrho(\mathrm{Im}[\pi_1(Z^{\mathrm{norm}}) \to \pi_1(X)]) = \{1\}$. Therefore, $\varrho(\mathrm{Im}[\pi_1(W^{\mathrm{norm}}) \to \pi_1(Y)])$ is finite. This contradicts with out assumption that $\mathfrak{C}$ is large. Hence, f is a one-to-one proper holomorphic map of complex normal spaces. Consequently, it is biholomorphic. □

The rest of the proof is same as in Theorem 4.26. By the proof of Theorem 4.25, there exist

— a topological Galois unramified covering $q : \widetilde{S}_H \to W$, where W is a projective normal variety;
— a positive closed $(1,1)$-current with continuous potential T_0 over W such that $\{T_0\}$ is Kähler;
— a continuous semi-positive plurisubharmonic function ϕ on $\widetilde{S}_H$ such that we have

$$\mathrm{dd}^{\mathrm{c}}\phi \geq q^*T_0. \tag{A.1}$$

By Claim A.6, $\widetilde{Y}_{\mathfrak{C}}$ can be identified with $\widetilde{S}_H$. Let $p : \widetilde{Y}' \to \widetilde{Y}_{\mathfrak{C}}$ be the intermediate Galois covering of Y between $\widetilde{Y} \to \widetilde{Y}_{\mathfrak{C}}$. By (A.1) we have

$$\mathrm{dd}^{\mathrm{c}}p^*\phi \geq (q \circ p)^*T_0. \tag{A.2}$$

We apply Proposition 1.14 to conclude that $\widetilde{Y}'$ is Stein. □

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Tuesday 6th June, 2023

YA DENG • *E-mail :* ya.deng@math.cnrs.fr, CNRS, Institut Élie Cartan de Lorraine, Université de Lorraine, F-54000 Nancy, France. • *Url :* https://ydeng.perso.math.cnrs.fr

KATSUTOSHI YAMANOI • *E-mail :* yamanoi@math.sci.osaka-u.ac.jp, Department of Mathematics, Graduate School of Science,Osaka University, Toyonaka, Osaka 560-0043, Japan
Url : https://sites.google.com/site/yamanoimath/